Servomotors are widely used electro-mechanical components. For example, they are used in steering systems for scale model helicopters and other radio controlled cars, robots, and aircraft, including military "drones".

A servomotor includes an electric motor, circuitry to control its speed and direction, and gearing to attain the high torques needed to apply moderately large forces over relatively short linear displacements. The gearing inside this servo is a compound gear train with four stages.

In this video, we will understand how linear algebra can help us to predict the losses and understand the design trade-offs when converting high speeds and low torques into low speeds and high torques using a gear train.

This video is part of the Linearity Video Series. Many complex systems are modeled or approximated linearly because of the mathematical advantages.

Hi, my name is Dan Frey, and I am a professor of Mechanical Engineering and Engineering Systems at MIT.

Before watching this video, you should be able to identify and describe the forces and torques that act on a system.

After watching this video and some practice, you will be able to: Model the forces and torques in a gear train as a system of linear equations.

Combine these linear equations into a matrix equation.

First, let's get a quick overview of the design. Here's a servomotor with the case opened.

Inside the case is a DC motor with a small pinion gear on it. The pinion has 10 teeth and is mated to a gear with 72 teeth. So the first pair provides a speed reduction of 7.2:1.

This large, 72 tooth gear is part of a single molding with a 10 tooth gear.

The 10 tooth gear on top of the 72 tooth gear is mated to a plastic gear with 48 teeth as you see here. So that stage provides a speed reduction of 4.8:1. The 48 tooth gear is molded to another 10 tooth gear.

The 11-tooth gear is mated to a black 36-tooth gear. This 36-tooth gear is molded to a 16-tooth gear. The 16-tooth gear is mated to this black 42 tooth gear, which is splined onto the output shaft.

The overall sequence of gear pairs is: 10 teeth mated to 72 teeth, 10 mated to 48, 11 mated to 36 and finally 16 mated to 42 teeth on the splined output shaft.

The overall gear ratio is the product of all the gear ratios of the gear pairs in the train.

Therefore about 326 to 1 as shown by this formula.

Based on the motor's output and some measurement and calculation, we would expect an output torque of 6.8 Newton meters in an idealized gear train.

But there are frictional losses at every stage of the power transmission process, so we guess the output shaft will provide substantially less torque than that. So let's start by measuring the maximum force at the output shaft, and convert that to an output torque.

We attach weights to a servo horn that has been connected with the output shaft on the servomotor. An electrical power source (in this case, a battery) is connected through a radio controller.

Here you see the servomotor lifting 1 kg. Here it is lifting about 2kg. And here it lifts even 3kg! This lightweight (100 gram) servomotor produces large amounts of torque, which can apparently lift more than 30 times its own weight.

In this set-up, the servomotor will lift a 4.42kg weight starting from a 90 degree angle.

As the motor lifts the weight, the readout on the scale should change.

Here you see the initial weight, with slack in the string, is 9.75 lbs. This is about 4.42kg. With the motor pulling at maximum capacity, the scale reads 3.70 lbs, which is 1.68 kg.

A calculation tells us that the maximum torque available at the output shaft according to our tests is 1.47 Newton meters.

We can use linear algebra to work out a better estimate. The force transmission and frictional losses in each step of the train can be modeled by a set of linear equations, (summing forces and summing torques on the rigid body). For the three rigid bodies that are comprised of two gears molded together, we will need three equations (sums of separate x and y forces and sums of torques).

The pinion gear and last gear in the train are simpler, because there is only one gear in the body, so we can combine each of those into a single equation.

The equations modeling the entire train can be assembled by linking together the five sets of equations with four additional equations to link each mating pair, so the overall system will have 15 equations--- 3 sets of 3 each (9 total) plus 1 each for the input and output gears plus 4 equations that model the connections between the mating gears. The solution to that set of equations helps us to estimate performance of the machine and also gives us insight into its design.

Let's make the model of just one gear body now, say the 10 tooth and 48 tooth molded gear seen here. One way to model the system is to posit that there are three unknown forces acting on this single body of two gears molded together.

We can name these forces F-TL2, F-TS2 , F-ShaftX2, and F-ShaftY2. The force F-TL2 is a force tangent to the gear pitch circle on the larger of the two gears. The T is for Tangent, the L is for Large, and the 2 indicates that this is the second compound gear in the train.

The force F-TS2 is a force tangent to the gear pitch circle on the smaller of the two gears. Again the T means tangent, and the S here means small.

Recall that these tangential forces really come from forces normal to the gear teeth.

These gears are designed so that all of the gears in this servomotor have a pressure angle of 20 degrees. So there is a X component to this force, the separation force between the mating gears, which has magnitude given by FTL2 and FTS2 times tan 0.35, where 0.35 is 20 degrees expressed in radians.

The forces F-ShaftX2 and F-ShaftY2 are the X and Y components of the normal force to the shaft, which is applied to the gear it supports.

There are also frictional forces associated with the normal force supporting the shaft, but they are not separate unknowns, they link the X and Y components normal forces by the friction coefficient, mu2.

This completes the description of the unknown forces in our model. Those four forces appear in three different linear equations as represented by this matrix: Take a moment to verify that these equations balance the forces and torques.

The first row in the matrix represents a sum of the forces on the gear in the X direction.

The second row represents a sum of the forces on the gear in the Y direction. The third row represents a sum of the torques or moments on the gear in the direction parallel to the gear shaft.

Now let's work on the model of the next gear in the train -- the one with 16 teeth and 36 teeth molded together. Again, we model the system by positing that there are four unknown forces acting on this single body having two gears molded together and we can name these forces FTL3, FTS3 , FShaftX3, and FShaftY3. And we obtain this matrix.

How can we link these two matrices together to get a model of the second and third gears combined? We see that the smaller gear on body 2 is in contact with the larger gear on body 3. According to Newton's third law, the reaction force on body 2 should be equal and opposite to that on body 3.

In fact, the situation is more complex. As the gears enter mating they slide into engagement.

As the gears depart contact, they slide back out. There are losses at the interface that are complex to model, and we will represent them simply using the efficiency in the average transmitted force.

Now we can join the model of the second gear and the third gear into a single system of linear equations represented by this matrix: The row in the middle is like "glue" holding the two models together. The force on the large gear on body 3 is equal to the force on the large gear on body 2 except a penalty is applied for inefficiencies. The principal mechanism for loss of power and torque is sliding friction -- in this case, shearing the lubricant on the gear faces.

Before going on to assemble more of the model, it is worth inspecting our work so far for patterns. Note that the top right and bottom left of our matrices are filled with 3x4 matrices of zeros. That is a clue that our matrix is shaping up to be banded in structure. Although every variable affects every other variable, the influences propagate locally (in some sense). For example, the frictional losses on shaft 2 do influence the frictional losses on shaft 3, but only indirectly through their effect on the force between the two bodies.

Because we arranged the variables in our vectors in a way that respects this structure, our matrix has a band in the middle from the top left corner to the bottom right. We will try to keep this up as we continue building the model. It will clarify interpretation of the model and also aid in efficiency and stability of the computations needed to solve it.

Now it's your turn, take a moment to write a matrix that models the force and torque balance on the first rigid body gear in the gear train.

This matrix is completely analogous to the matrices we obtained in for the 2nd and 3rd gear bodies.

When we add the pinion gear and fourth gear into the model, we need to take additional care. These will have slightly different structure. The gear connected to the pinion on the motor has an applied torque due to the motor. The reaction at the output shaft is an externally applied torque, due entirely to the large fourth gear. It is possible to combine these into a single equation for each gear.

Pause the video and try to write out the full system of linear equations for the overall 4 gear train system.

Placing these into the model as the first and last rows, the overall system of linear equations is: Note that the matrix has 15 rows and 15 columns. This should be expected as the system should be neither overdetermined nor underdetermined. Given a particular torque input at the motor, there should be a single value of output torque consistent with our gear train model and its deterministically defined parameters.

By solving this system, we find unique values for all the unknowns. When we assembled the equations, we guessed the direction of each force in the free body diagram. So, it's useful to inspect the solution for negative values. For the parameter values we chose, the reaction force at the shaft of the third body in the x direction turns out to be negative.

We should ask ourselves, did we guess wrong about the direction of the force, or is there something wrong with our matrix? Looking at the forces computed it seems that the friction in the x direction due to support of the large y reaction on this gear was more dominant that we expected.

It's sensible to run some sanity checks in a case like this. If the friction on that gear were very low, would the sign be positive as expected? We ran that scenario, and the sign did become positive. So our guess about the direction of the reaction force was wrong.

Our assumptions for the magnitude of the friction were too low. But it seems the model is behaving in reasonable ways.

Now that we have a reasonable level of confidence in the model, we can begin to use it to explore the design decisions that the engineers made. If we put in reasonable values for the gear mating efficiency, such as 96%, and reasonable values for the friction coefficients at the bushings, such as 0.3, we find an overall efficiency of the gear train is about 52%.

This is in reasonable agreement with our simple measurements. We found that the maximum torque available at the output shaft is 1.47 Newton meters, which would imply an efficiency closer to 25%.

But our measurement used an overhanging load, which caused the shaft to bend. Our simple model makes a large and optimistic assumption of loading the servomotor with torques only.

It's not surprising that the answers differ substantially. It suggests installing the servo so that bending loads are supported elsewhere, not in the servo itself.

The gears in automotive transmissions with similar ratios are much more efficient, perhaps 90% to 95% efficient. But for a compact and inexpensive gear train, this design performs well, especially since the plastic the gears are molded from adds a great deal of rolling friction.

To summarize, we built an engineering model of a servomotor gear train using systems of linear equations. The matrix representations helped us to explore the interactions among variables in the system like separation forces between gears and friction at the shafts.

Since the 15 by 15 system of equations is so fast and easy to solve on a modern computer, we could run a large number of "what if" scenarios.

We hope this video helped you see how linear algebra can be used to make and understand engineering design decisions.