21

Continuous-Time Second-Order Systems

The properties of the Laplace transform make it particularly useful in analyzing LTI systems that are represented by linear constant-coefficient differential equations. Specifically, applying the Laplace transform to a differential equation converts it to an algebraic equation relating the Laplace transform of the system output to the product of the Laplace transform of the system input and the Laplace transform of the system impulse response, referred to as the *system function*. The system function is readily obtained by inspection of the differential equation, and the system impulse response can be obtained by evaluating the inverse Laplace transform of the system function. Alternatively, the response for any other input can be evaluated by first multiplying the Laplace transform of the input by the system function and then applying the inverse Laplace transform.

Two particularly important classes of systems described by linear constant-coefficient differential equations are first-order and second-order systems. In implementing higher-order systems, it is very common to use first- and second-order systems as building blocks. Much of this lecture focuses on using the Laplace transform to describe the behavior of these building blocks.

First-order systems are represented by a single pole in the s-plane, and second-order systems by a pair of poles. There may or may not also be zeros in the transfer function, depending on whether there are derivative terms on the right-hand side of the differential equation. From the differential equation, the system function can be written directly. If we assume that the systems are causal, so that the impulse response is right-sided, then the ROC of the system function is implicitly specified to be to the right of the rightmost pole in the s-plane.

For second-order systems, the poles may be either on the real axis in the s-plane or off the real axis as a complex conjugate pair, depending on the specific relationship between the coefficients. When both poles are real-valued, the system is often referred to as *overdamped*, and when they occur as a complex-conjugate pair the system is referred to as *underdamped*. In the time domain, the underdamped case corresponds to an oscillatory impulse response with an exponential damping. The time constant of the damping is related to the real part of the pole locations, and the oscillatory behavior is associated with the imaginary part. As the poles move closer to the $j\omega$-axis the damping decreases, and as the poles move parallel to the $j\omega$-axis the oscillatory behavior changes in frequency.

Many of the properties of the frequency response of a system can be inferred from inspection of the pole-zero pattern of the system function. Since the Laplace transform reduces to the Fourier transform for $s = j\omega$, the behavior of the system function on the $j\omega$-axis corresponds to the system frequency response. By considering the behavior of the associated vectors in the s-plane, we can infer the behavior of the frequency response for underdamped second-order systems. In particular, the frequency response tends to have a peak for the underdamped case, and as the poles move closer to the $j\omega$-axis this peak becomes increasingly sharp. The frequency location of this peak or resonance is closely associated with the frequency of oscillation of the impulse response, and the width of the peak is closely associated with the damping of the oscillations.

Since higher-order transfer functions can always be decomposed into a product or sum of first-order and second-order transfer functions, these are important building blocks for more general systems. One illustration of this is the use of second-order systems in speech synthesis. The use of second-order underdamped systems to simulate the resonances of the vocal tract for generating synthesized speech is discussed and illustrated in this lecture.

Suggested Reading

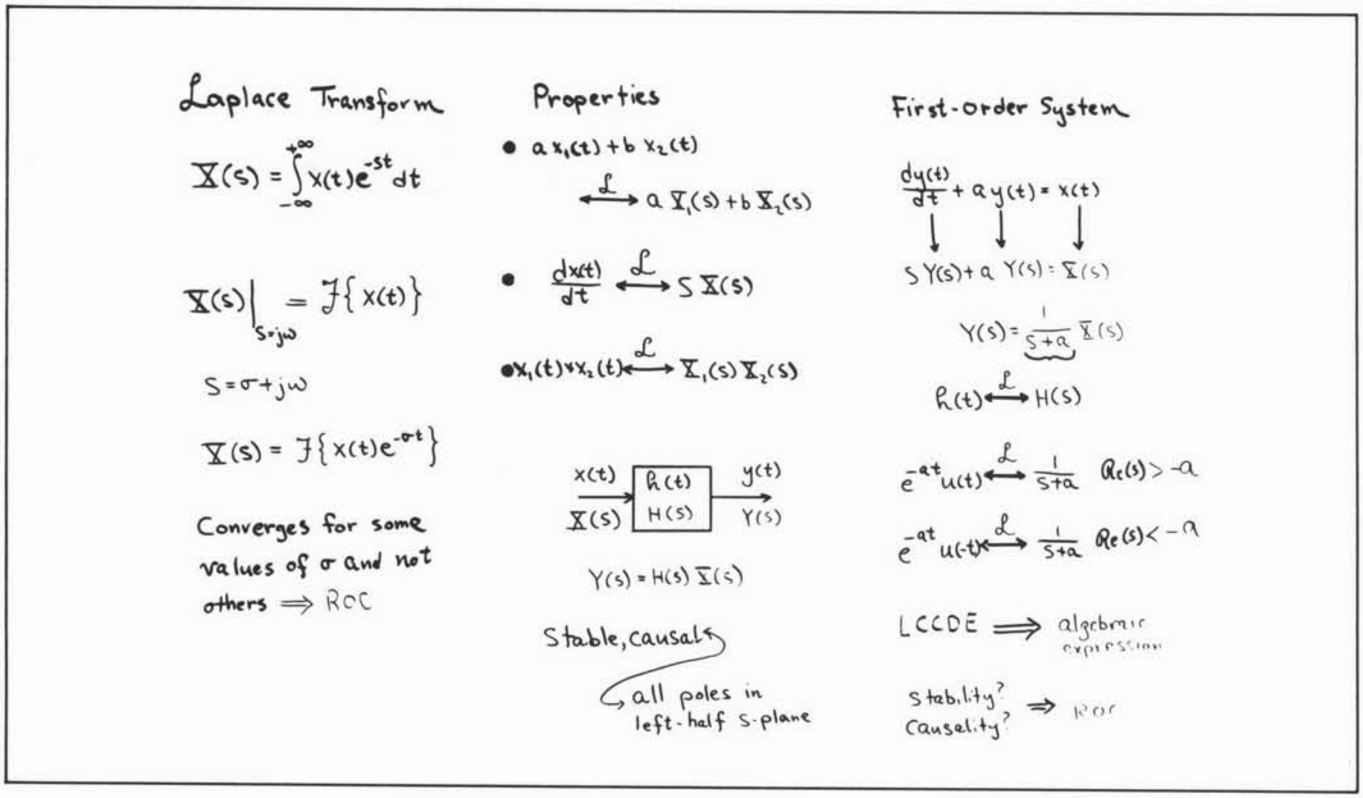
Section 9.5, Properties of the Laplace Transform, pages 596–603

Section 9.7, Analysis and Characterization of LTI Systems Using the Laplace Transform, pages 604–611

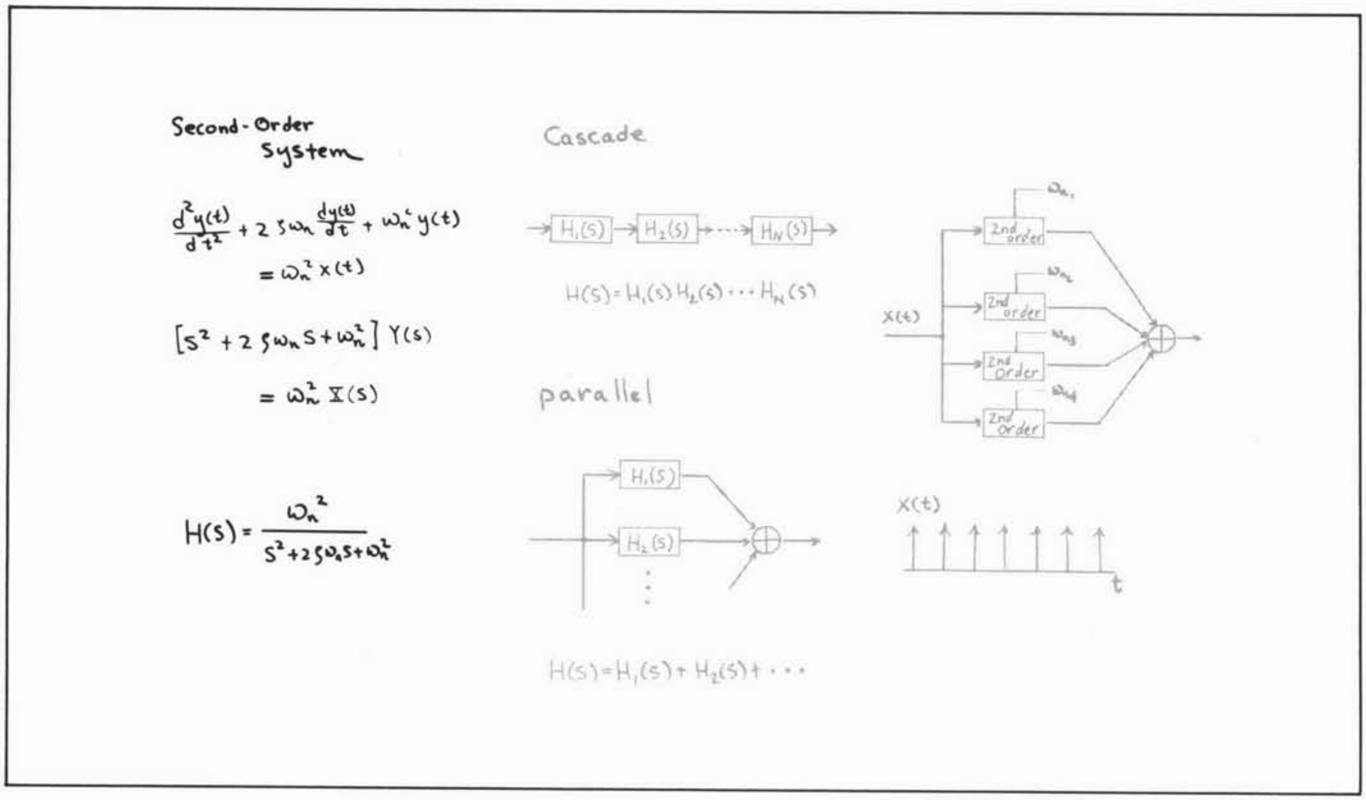
Section 4.12, First-Order and Second-Order Systems, pages 240–250

Section 9.4, Geometric Evaluation of the Fourier Transform from the Pole-Zero Plot, pages 590–595

MARKERBOARD
21.1



MARKERBOARD
21.2 (a)



TRANSPARENCY 21.1
System function for a second-order system.

SECOND-ORDER SYSTEM

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$H(s) = \frac{\omega_n^2}{(s-c_1)(s-c_2)}$$

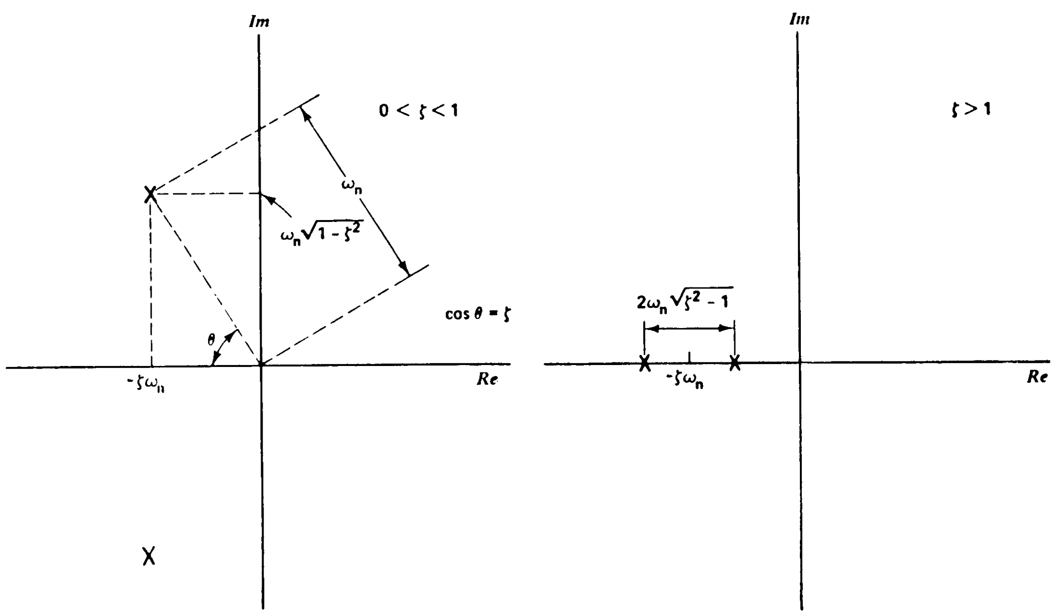
$$c_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}$$

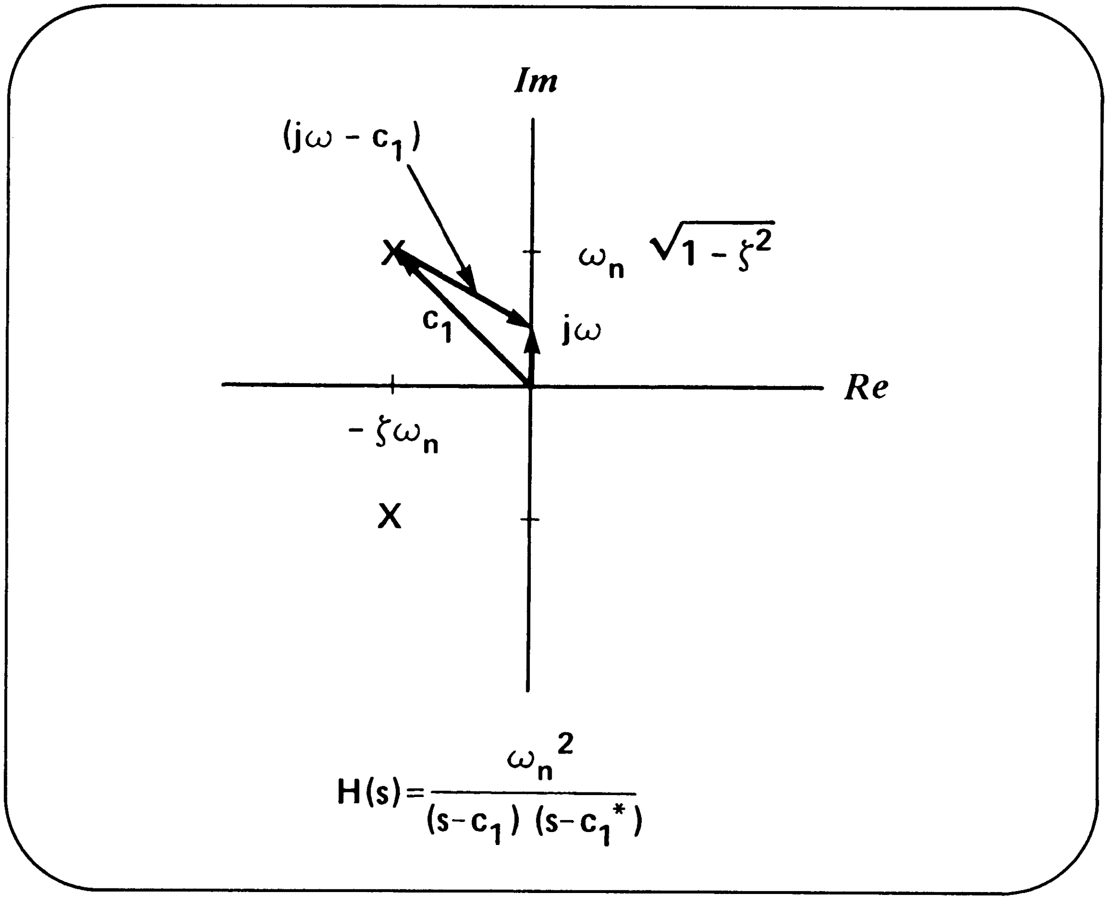
$$c_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$$

For $\zeta < 1$, $c_1 = c_2^*$

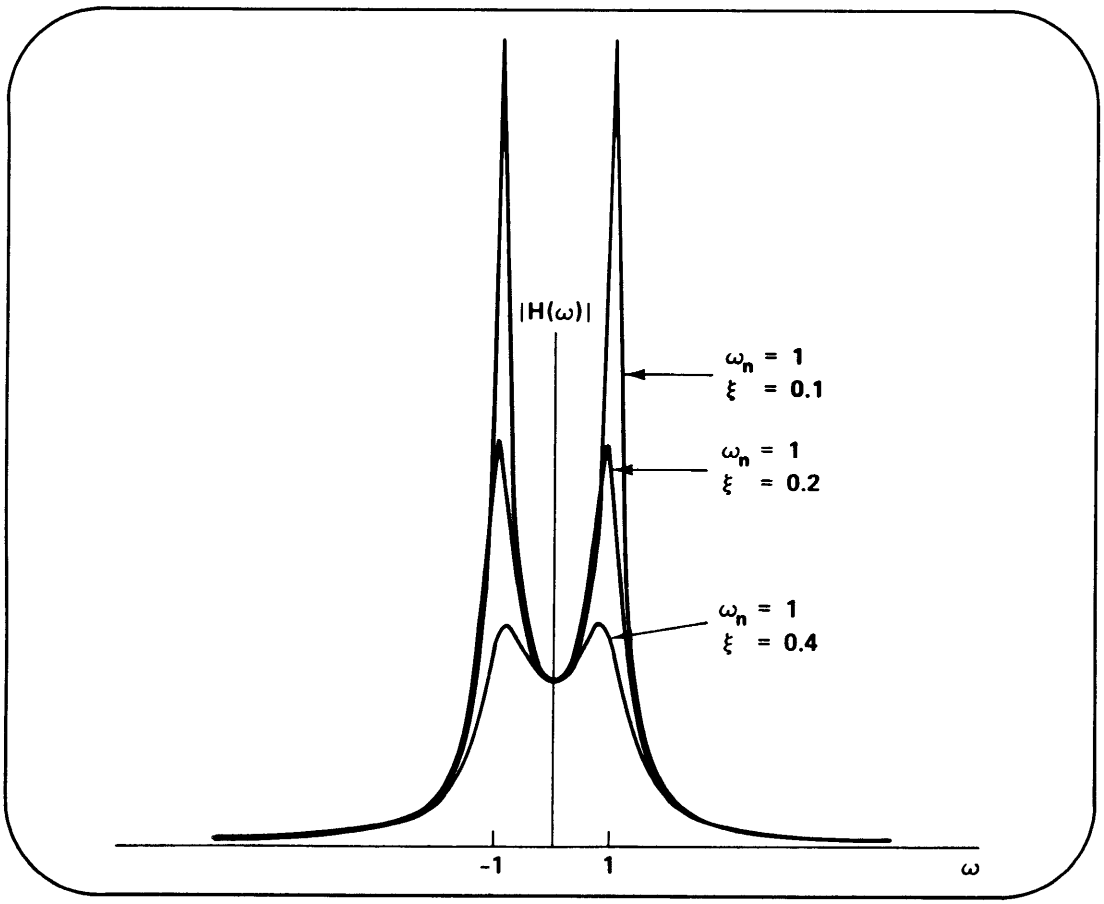
$$= -\zeta\omega_n + j\omega_n\sqrt{1 - \zeta^2}$$

TRANSPARENCY 21.2
Pole-zero pattern associated with an underdamped (left) and with an overdamped (right) second-order system.





TRANSPARENCY
21.3
Determination of the frequency response of a second-order system from the pole-zero pattern.



TRANSPARENCY
21.4
Frequency response for an underdamped second-order system.

TRANSPARENCY 21.5
System function for a second-order system. [Transparency 21.1 repeated]

SECOND-ORDER SYSTEM

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$H(s) = \frac{\omega_n^2}{(s-c_1)(s-c_2)}$$

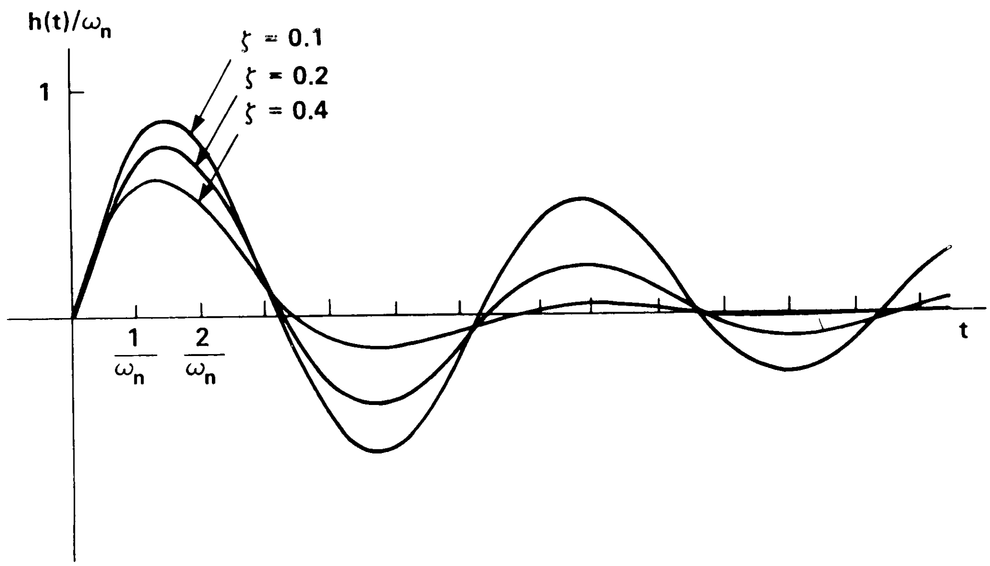
$$c_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}$$

$$c_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$$

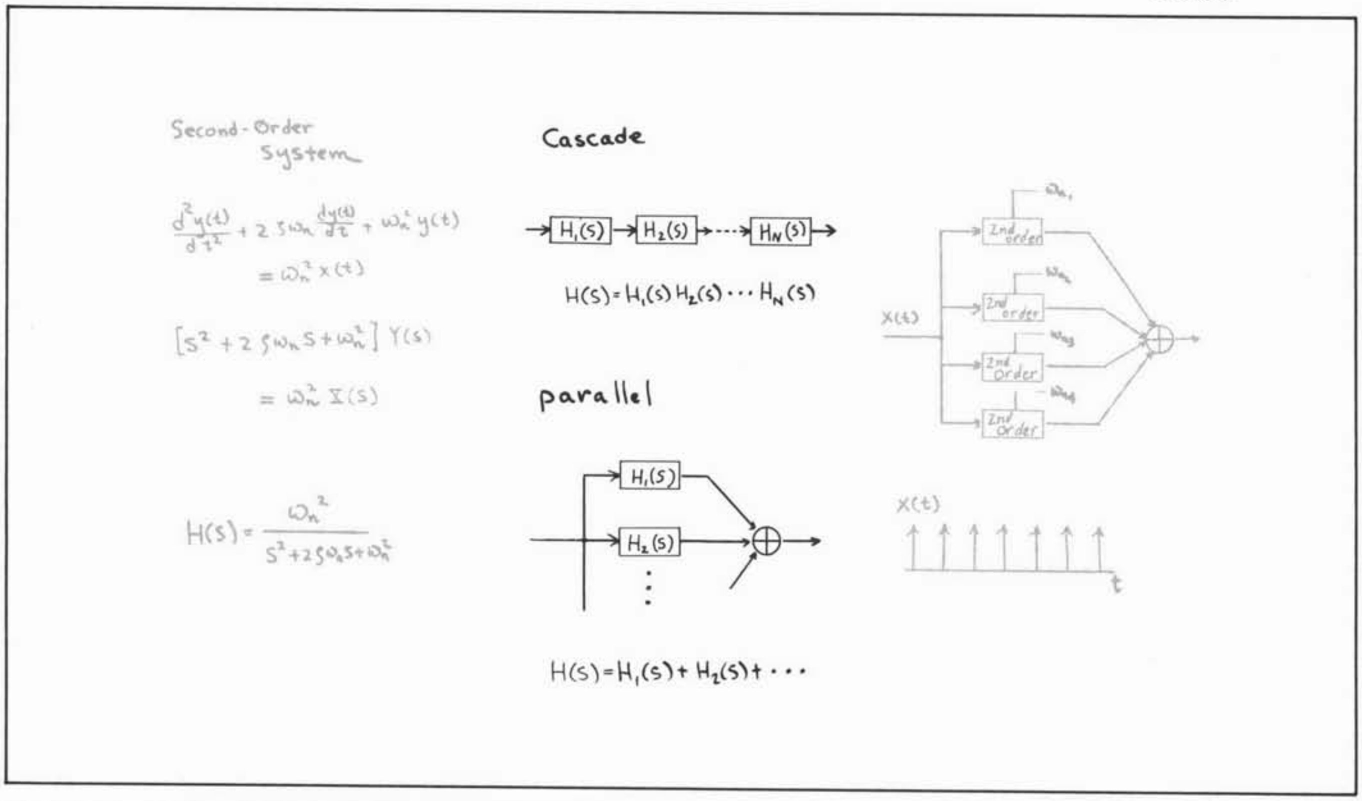
$$\text{For } \zeta < 1,\ c_1 = c_2^*$$

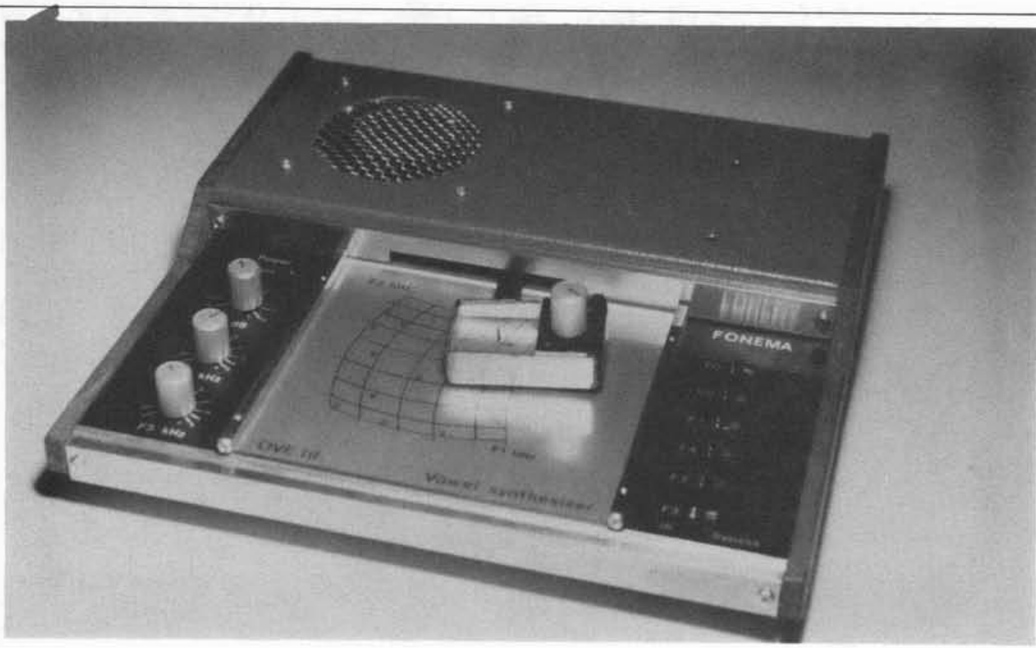
$$= -\zeta\omega_n + j\omega_n\sqrt{1 - \zeta^2}$$

TRANSPARENCY 21.6
Impulse response for an underdamped second-order system.



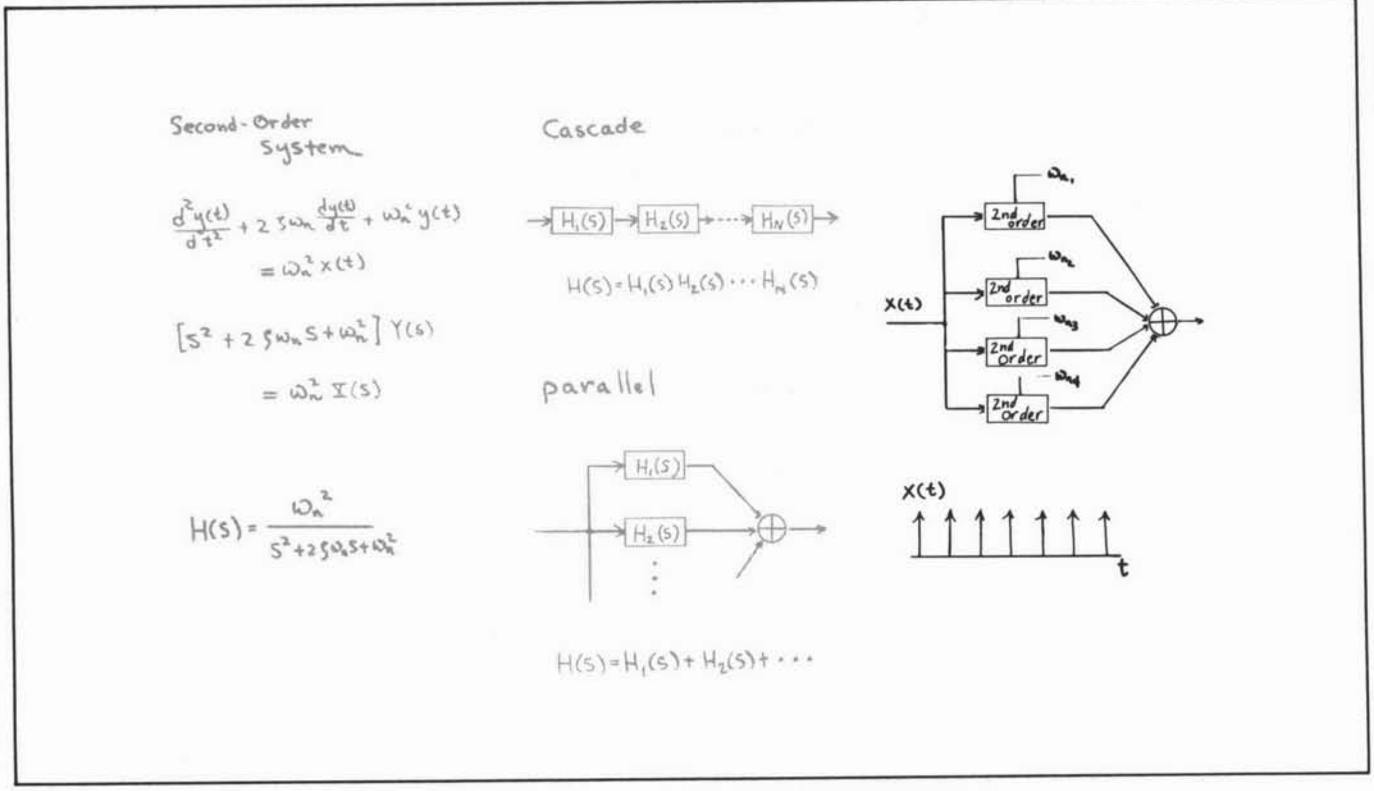
MARKERBOARD
21.2 (b)





DEMONSTRATION
21.1
Vowel synthesizer demonstrating the use of second-order continuous-time filters in speech synthesis.

MARKERBOARD
21.2 (c)



DEMONSTRATION 21.2
The Texas Instruments Speak & Spell, which uses discrete-time filters for speech synthesis.