PROFESSOR: In the last lecture, I introduced and illustrated the kinds of signals and systems that we'll be dealing with throughout this course. In today's lecture I'd like to be a little more specific, and in particular, talk about some of the basic signals, both continuous-time and discrete-time that will form important building blocks as the course progresses.

Let's begin with one signal, the continuous-time sinusoidal signal, which perhaps you're already somewhat familiar with. Mathematically, the continuous-time sinusoidal signal is expressed as I've indicated here. There are three parameters, A, omega_0 and phi. The parameter A is referred to as the amplitude, the parameter omega 0 as the frequency, and the parameter phi as the phase. And graphically, the continuous-time sinusoidal signal has the form shown here.

Now, the sinusoidal signal has a number of important properties that we'll find it convenient to exploit as the course goes along, one of which is the fact that the sinusoidal signal is what is referred to as periodic. What I mean by periodic is that under an appropriate time shift, which I indicate here as T_0, the signal replicates or repeats itself. Or said another way, if we shift the time origin by an appropriate amount T_0, the smallest value T_0 being referred to as the period, then x(t) is equal to itself, shifted.

And we can demonstrate it mathematically by simply substituting into the mathematical expression for the sinusoidal signal t + T_0, in place of t. When we carry out the expansion we then have, for the argument of the sinusoid, omega_0 t + omega_0 T_0 + phi.

Now, one of the things that we know about sinusoidal functions is that if you change the argument by any integer multiple of 2 pi, then the function has the same value. And so we can exploit that here, in particular with omega_0 T_0 and integer multiple of 2 pi. Then the right-hand side of this equation is equal to the left-hand side of the equation.

So with omega_0 T_0 equal to 2 pi times an integer, or T_0 equal to 2 pi times an integer divided by omega_0, the signal repeats. The period is defined as the smallest value of T_0. And so the period is 2 pi divided by omega_0.

And going back to our sinusoidal signal, we can see that-- and I've indicated here, then, the period as 2 pi / omega_0. And that's the value under which the signal repeats.

Now in addition, a useful property of the sinusoidal signal is the fact that a time shift of a sinusoid is equivalent to a phase change. And we can demonstrate that again mathematically, in particular if we put the sinusoidal signal under a time shift-- I've indicated the time shift that I'm talking about by t_0-- and expand this out, then we see that that is equivalent to a change in phase.

And an important thing to recognize about this statement is that not only is a time shift generating a phase change, but, in fact, if we inserted a phase change, there is always a value of t_0 which would correspond to an equivalent time shift. Said another way, if we take omega_0 t_0 and think of that as our change in phase, for any change in phase, we can solve this equation for a time shift, or conversely for any value of time shift, that represents an appropriate phase.

So a time shift corresponds to a phase change, and a phase change, likewise, corresponds to time shift. And so for example, if we look at the general sinusoidal signal that we saw previously, in effect, changing the phase corresponds to moving this signal in time one way or the other. For example, if we look at the sinusoidal signal with a phase equal to 0 that corresponds to locating the time origin at this peak. And I've indicated that on the following graph.

So here we have illustrated a sinusoid with 0 phase, or a cosine with 0 phase, corresponding to taking our general picture and shifting it. Shifting it appropriately as I've indicated here. This, of course, still has the property that it's a periodic function, since we simply displaced it in time.

And by looking at the graph, what we see is that it has another very important property, a property referred to as even. And that's a property that we'll find useful, in general, to refer to in relation to signals. A signal is said to be even if, when we reflect it about the origin, it looks exactly the same. So it's symmetric about the origin.

And looking at this sinusoid, that, in fact, has that property. And mathematically, the statement that it's even is equivalent to the statement that if we replace the time argument by its negative, the function itself doesn't change.

Now this corresponded to a phase shift of 0 in our original cosine expression. If instead, we had chosen a phase shift of, let's say, -pi/2, then instead of a cosinusoidal signal, what we would regenerate is a sinusoid with the appropriate phase. Or, said another way, if we take our original cosine and substitute in for the phase -pi/2, then of course we have this mathematical expression.

Using just straightforward trigonometric identities, we can express that alternately as sin(omega_0*t). The frequency and amplitude, of course, haven't changed. And that, you can convince yourself, also is equivalent to shifting the cosine by an amount in time that I've indicated here, namely a quarter of a period.

So illustrated below is the graph now, when we have a phase of -pi/2 in our cosine, which is a sinusoidal signal. Of course, it's still periodic. It's periodic with a period of 2 pi / omega_0 again, because all that we've done by introducing a phase change is introduced the time shift.

Now, when we look at the sinusoid in comparison with the cosine, namely with this particular choice of phase, this has a different symmetry, and that symmetry is referred to odd. What odd symmetry means, graphically, is that when we flip the signal about the time origin, we also multiply it by a minus sign. So that's, in effect, anti-symmetric. It's not the mirror image, but it's the mirror image flipped over.

And we'll find many occasions, not only to refer to signals more general than sinusoidal signals, as even in some cases and odd in other cases. And in general, mathematically, an odd signal is one which satisfies the algebraic expression, x(t). When you replace t by its negative, is equal to -x(-t). So replacing the argument by its negative corresponds to an algebraic sign reversal.

OK. So this is the class of continuous-time sinusoids. We'll have a little more to say about it later. But I'd now like to turn to discrete-time sinusoids. What we'll see is that discrete-time sinusoids are very much like continuous-time ones, but also with some very important differences. And we want to focus, not only on the similarities, but also on the differences.

Well, let's begin with the mathematical expression. A discrete-time sinusoidal signal, mathematically, is as I've indicated here, A cos(omega_0 n + phi). And just as in the continuous-time case, the parameter A is what we'll refer to as the amplitude, omega_0 as the frequency, and phi as the phase.

And I've illustrated here several discrete-time sinusoidal signals. And they kind of look similar. In fact, if you track what you might think of as the envelope, it looks very much like what a continuous-time sinusoid might look like. But keep in mind that the independent variable, in this case, is an integer variable. And so the sequence only takes on values at integer values of the argument. And we'll see that has a very important implication, and we'll see that shortly.

Now, one of the issues that we addressed in the continuous-time case was periodicity. And I want to return to that shortly, because that is one of the areas where there is an important distinction. Let's first, though, examine the statement similar to the one that we examined for continuous time, namely the relationship between a time shift and a phase change.

Now, in continuous time, of course, we saw that a time shift corresponds to a phase change, and vice versa. Let's first look at the relationship between shifting time and generating a change in phase. In particular for discrete time, if I implement a time shift that generates a phase change-- and we can see that easily by simply inserting a time shift, n + n_0. And if we expand out this argument, we have omega_0 n + omega_0 n_0.

And so I've done that on the right-hand side of the equation here. And the omega_0 n_0, then, simply corresponds to a change in phase. So clearly, a shift in time generates a change in phase.

And for example, if we take a particular sinusoidal signal, let's say we take the cosine signal at a particular frequency, and with a phase equal to 0, a sequence that we might generate is one that I've illustrated here. So what I'm illustrating here is the cosine signal with 0 phase. And it has a particular behavior to it, which will depend somewhat on the frequency.

If I now take this same sequence and shift it so that the time origin is shifted a quarter of a period away, then you can convince yourself-- and it's straightforward to work out-- that that time shift corresponds to a phase shift of pi/2. So in that case, with the cosine with a phase of -pi/2, that will correspond to the expression that I have here.

We could alternately write that, using again a trigonometric identity, as a sine function. And that, I've stated, is equivalent to a time shift. Namely, this shift of pi/2 is equal to a certain time shift, and the time shift for this particular example is a quarter of a period.

So here, we have the sinusoid. Previously we had the cosine. The cosine was exactly the same sequence, but with the origin located here. And in fact, that's exactly the way we drew this graph. Namely, we just simply took the same values and changed the time origin.

Now, looking at this sequence, which is the sinusoidal sequence, the phase of -pi/2, that has a certain symmetry. And in fact, what we see is that it has an odd symmetry, just as in the continuous-time case. Namely, if we take that sequence, flip it about the axis, and flip it over in sign, that we get the same sequence back again. Whereas with 0 phase corresponding to the cosine that I showed previously, that has an even symmetry. Namely, if I flip it about the time origin and don't do a sign reversal, then the sequence is maintained.

So here, we have an odd symmetry, expressed mathematically as I've indicated. Namely, replacing the independent variable by its negative attaches a negative sign to the whole sequence. Whereas in the previous case, what we have is 0 phase and an even symmetry. And that's expressed mathematically as x[n] = x[-n].

Now, one of the things I've said so far about discrete-time sinusoids is that a time shift corresponds to a phase change. And we can then ask whether the reverse statement is also true, and we knew that the reverse statement was true in continuous time. Specifically, is it true that a phase change always corresponds to a time shift? Now, we know that that is true, namely, that this statement works both ways in continuous time. Does it in discrete time?

Well, the answer, somewhat interestingly or surprisingly until you sit down and think about it, is no. It is not necessarily true in discrete time that any phase change can be interpreted as a simple time shift of the sequence. And let me just indicate what the problem is.

If we look at the relationship between the left side and the right side of this equation, expanding this out as we did previously, we have that omega_0 n + omega_0 n_0 must correspond to omega_0 n + phi. And so omega_0 n_0 must correspond to the phase change. Now, what you can see pretty clearly is that depending on the relationship between phi and omega_0, n_0 may or may not come out to be an integer.

Now, in continuous time, the amount of time shift did not have to be an integer amount. In discrete time, when we talk about a time shift, the amount of time shift-- obviously, because of the nature of discrete time signals-- must be an integer. So the phase changes related to time shifts must satisfy this particular relationship. Namely, that omega_0 n_0, where n_0 is an integer, is equal to the change in phase.

OK. Now, that's one distinction between continuous time and discrete time. Let's now focus on another one, namely the issue of periodicity. And what we'll see is that again, whereas in continuous time, all continuous-time sinusoids are periodic, in the discrete-time case that is not necessarily true.

To explore that a little more carefully, let's look at the expression, again, for a general sinusoidal signal with an arbitrary amplitude, frequency, and phase. And for this to be periodic, what we require is that there be some value, N, under which, when we shift the sequence by that amount, we get the same sequence back again. And the smallest-value N is what we've defined as the period.

Now, when we try that on a sinusoid, we of course substitute in for n, n + N. And when we expand out the argument here, we'll get the argument that I have on the right-hand side. And in order for this to repeat, in other words, in order for us to discard this term, omega_0 N, where N is the period, must be an integer multiple of 2 pi. And in that case, it's periodic as long as omega_0 N, N being the period, is 2 pi times an integer. Just simply dividing this out, we have N, the period, is 2 pi m / omega_0.

Well, you could say, OK what's the big deal? Whatever N happens to come out to be when we do that little bit of algebra, that's the period. But in fact, N, or 2 pi m / omega_0, may not ever come out to be an integer. Or it may not come out to be the one that you thought it might.

For example, let's look at some particular sinusoidal signals. Let's see. We have the first one here, which is a sinusoid, as I've shown. And it has a frequency, what I've referred to as the frequency, omega_0 = 2 pi / 12. And what we'd like to look at is 2 pi / omega_0, then find an integer to multiply that by in order to get another integer.

Let's just try that here. If we look at 2 pi / omega_0, 2 pi / omega_0, for this case, is equal to 12. Well, that's fine. 12 is an integer. So what that says is that this sinusoidal signal is periodic. And in fact, it's periodic with a period of 12.

Let's look at the next one. The next one, we would have 2 pi / omega_0 again. And that's equal to 31/4. So what that says is that the period is 31/4. But wait a minute. 31/4 isn't an integer. We have to multiply that by an integer to get another integer. Well, we'd multiply that by 4, so (2 pi / omega_0) times 4 is 31, 31 is an integer. And so what that says is this is periodic, not with a period of 2 pi / omega_0, but with a period of (2 pi / omega_0) times 4, namely with a period of 31.

Finally, let's take the example where omega_0 is equal to 1/6, as I've shown here. That actually looks, if you track it with your eye, like it's periodic. 2 pi / omega_0, in that case, is equal to 12 pi. Well, what integer can I multiply 12 pi by and get another integer? The answer is none, because pi is an irrational number.

So in fact, what that says is that if you look at this sinusoidal signal, it's not periodic at all, even though you might fool yourself into thinking it is simply because the envelope looks periodic. Namely, the continuous-time equivalent of this is periodic, the discrete-time sequence is not.

OK. Well, we've seen, then, some important distinctions between continuous-time sinusoidal signals and discrete-time sinusoidal signals. The first one is the fact that in the continuous-time case, a time shift and phase change are always equivalent. Whereas in the discrete-time case, in effect, it works one way but not the other way.

We've also seen that for a continuous-time signal, the continuous-time signal is always periodic, whereas the discrete-time signal is not necessarily. In particular, for the continuous-time case, if we have a general expression for the sinusoidal signal that I've indicated here, that's periodic for any choice of omega_0. Whereas in the discrete-time case, it's periodic only if 2 pi / omega_0 can be multiplied by an integer to get another integer.

Now, another important and, as it turns out, useful distinction between the continuous-time and discrete-time case is the fact that in the discrete-time case, as we vary what I've called the frequency omega_0, we only see distinct signals as omega_0 varies over a 2 pi interval. And if we let omega_0 vary outside the range of, let's say, -pi to pi, or 0 to 2 pi, we'll see the same sequences all over again, even though at first glance, the mathematical expression might look different.

So in the discrete-time case, this class of signals is identical for values of omega_0 separated by 2 pi, whereas in the continuous-time case, that is not true. In particular, if I consider these sinusoidal continuous-time signals, as I vary omega_0, what will happen is that I will always see different sinusoidal signals. Namely, these won't be equal.

And in effect, we can justify that statement algebraically. And I won't take the time to do it carefully. But let's look, first of all, at the discrete-time case. And the statement that I'm making is that if I have two discrete-time sinusoidal signals at two different frequencies, and if these frequencies are separated by an integer multiple of 2 pi-- namely if omega_2 is equal to omega_1 + 2 pi times an integer m-- when I substitute this into this expression, because of the fact that n is also an integer, I'll have m * n as an integer multiple of 2 pi. And that term, of course, will disappear because of the periodicity of the sinusoid, and these two sequences will be equal.

On the other hand in the continuous-time case, since t is not restricted to be an integer variable, for different values of omega_1 and omega_2, these sinusoidal signals will always be different.

OK. Now, many of the issues that I've raised so far, in relation to sinusoidal signals, are elaborated on in more detail in the text. And of course, you'll have an opportunity to exercise some of this as you work through the video course manual. Let me stress that sinusoidal signals will play an extremely important role for us as building blocks for general signals and descriptions of systems, and leads to the whole concept Fourier analysis, which is very heavily exploited throughout the course.

What I'd now like to turn to is another class of important building blocks. And in fact, we'll see that under certain conditions, these relate strongly to sinusoidal signals, namely the class of real and complex exponentials.

Let me begin, first of all, with the real exponential, and in particular, in the continuous-time case. A real continuous-time exponential is mathematically expressed, as I indicate here, x(t) = C e ^ (a t), where for the real exponential, C and a are real numbers. And that's what we mean by the real exponential. Shortly, we'll also consider complex exponentials, where these numbers can then become complex.

So this is an exponential function. And for example, if the parameter a is positive, that means that we have a growing exponential function. If the parameter a is negative, then that means that we have a decaying exponential function.

Now, somewhat as an aside, it's kind of interesting to note that for exponentials, a time shift corresponds to a scale change, which is somewhat different than what happens with sinusoids. In the sinusoidal case, we saw that a time shift corresponded to a phase change. With the real exponential, a time shift, as it turns out, corresponds to simply changing the scale.

There's nothing particularly crucial or exciting about that. And in fact, perhaps stressing it is a little misleading. For general functions, of course, about all that you can say about what happens when you implement a time shift is that it implements a time shift.

OK. So here's the real exponential. Just C e ^ (a t). Let's look at the real exponential, now, in the discrete-time case. And in the discrete-time case, we have several alternate ways of expressing it. We can express the real exponential in the form C e ^ (beta n), or as we'll find more convenient, in part for a reason at I'll indicate shortly, we can rewrite this as C alpha ^ n, where of course, alpha = e ^ beta. More typically in the discrete-time case, we'll express the exponential as C alpha ^ n.

So for example, this becomes, essentially, a geometric series or progression as n continues for certain values of alpha. Here for example, we have for alpha greater than 0, first of all on the top, the case where the magnitude of alpha is greater than 1, so that the sequence is exponentially or geometrically growing. On the bottom, again with alpha positive, but now with its magnitude less than 1, we have a geometric progression that is exponentially or geometrically decaying.

OK. So this, in both of these cases, is with alpha greater than 0. Now the function that we're talking about is alpha ^ n. And of course, what you can see is that if alpha is negative instead of positive, then when n is even, that minus sign is going to disappear. When n is odd, there will be a minus sign. And so for alpha negative, the sequence is going to alternate positive and negative values.

So for example, here we have alpha negative, with its magnitude less than 1. And you can see that, again, its envelope decays geometrically, and the values alternate in sign. And here we have the magnitude of alpha greater than 1, with alpha negative. Again, they alternate in sign, and of course it's growing geometrically.

Now, if you think about alpha positive and go back to the expression that I have at the top, namely C alpha ^ n. With alpha positive, you can see a straightforward relationship between alpha and beta. Namely, beta is the natural logarithm of alpha.

Something to think about is what happens if alpha is negative? Which is, of course, a very important and useful class of real discrete-time exponentials also. Well, it turns out that with alpha negative, if you try to express it as C e ^ (beta n), then beta comes out to be an imaginary number. And that is one, but not the only reason why, in the discrete-time case, it's often most convenient to phrase real exponentials in the form alpha ^ n, rather than e ^ (beta n). In other words, to express them in this form rather than in this form.

Those are real exponentials, continuous-time and discrete-time. Now let's look at the continuous-time complex exponential. And what I mean by a complex exponential, again, is an exponential of the form C e ^ (a t). But in this case, we allow the parameters C and a to be complex numbers.

And let's just track this through algebraically. If C and *a* are complex numbers, let's write C in polar form, so it has a magnitude and an angle. Let's write *a* in rectangular form, so it has a real part and an imaginary part. And when we substitute these two in here, combine some things together-- well actually, I haven't combined yet. I have this for the amplitude factor, and this for the exponential factor. I can now pull out of this the term corresponding to e ^ (r t), and combine the imaginary parts together. And I come down to the expression that I have here.

So following this further, an exponential of this form, e ^ (j omega) or e ^ (j phi), using Euler's relation, can be expressed as the sum of a cosine plus j times a sine. And so that corresponds to this factor. And then there is this time-varying amplitude factor on top of it.

Finally putting those together, we end up with the expression that I show on the bottom. And what this corresponds to are two sinusoidal signals, 90 degrees out of phase, as indicated by the fact that there's a cosine and a sine. So there's a real part and an imaginary part, with sinusoidal components 90 degrees out of phase, and a time-varying amplitude factor, which is a real exponential. So it's a sinusoid multiplied by a real exponential in both the real part and the imaginary part.

And let's just see what one of those terms might look like. What I've indicated at the top is a sinusoidal signal with a time-varying exponential envelope, or an envelope which is a real exponential, and in particular which is growing, namely with r greater than 0. And on the bottom, I've indicated the same thing with r less than 0.

And this kind of sinusoidal signal, by the way, is typically referred to as a damped sinusoid. So with r negative, what we have in the real and imaginary parts are damped sinusoids. And the sinusoidal components of that are 90 degrees out of phase, in the real part and in the imaginary part.

OK. Now, in the discrete-time case, we have more or less the same kind of outcome. In particular we'll make reference to our complex exponentials in the discrete-time case. The expression for the complex exponential looks very much like the expression for the real exponential, except that now we have complex factors. So C and alpha are complex numbers.

And again, if we track through the algebra, and get to a point where we have a real exponential multiplied by a factor which is a purely imaginary exponential, apply

Euler's relationship to this, we then finally come down to a sequence, which has a real exponential amplitude multiplying one sinusoid in the real part. And in the imaginary part, exactly the same kind of exponential multiplying a sinusoid that's 90 degrees out of phase from that.

And so if we look at what one of these factors might look like, it's what we would expect given the analogy with the continuous-time case. Namely, it's a sinusoidal sequence with a real exponential envelope. In the case where alpha is positive, then it's a growing envelope. In the case where alpha is negative-- I'm sorry-- where the magnitude of alpha is greater than 1, it's a growing exponential envelope. Where the magnitude of alpha is less than 1, it's a decaying exponential envelope.

And so I've illustrated that here. Here we have the magnitude of alpha greater than 1. And here we have the magnitude of alpha less than 1. In both cases, sinusoidal sequences underneath the envelope, and then an envelope that is dictated by what the magnitude of alpha is.

OK. Now, in the discrete-time case, then, we have results similar to the continuous-time case. Namely, components in a real and imaginary part that have a real exponential factor times a sinusoid. Of course, if the magnitude of alpha is equal to 1, then this factor disappears, or is equal to 1. And this factor is equal to 1. And so we have sinusoids in both the real and imaginary parts.

Now, one can ask whether, in general, the complex exponential with the magnitude of alpha equal to 1 is periodic or not periodic. And the clue to that can be inferred by examining this expression. In particular, in the discrete-time case with the magnitude of alpha equal to 1, we have pure sinusoids in the real part and the imaginary part. And in fact, in a continuous-time case with r equal to 0, we have sinusoids in the real part and the imaginary part.

In a continuous-time case when we have a pure complex exponential, so that the terms aren't exponentially growing or decaying, those exponentials are always periodic. Because, of course, the real and imaginary sinusoidal components are periodic. In the discrete-time case, we know that the sinusoids may or may not be periodic, depending on the value of omega_0. And so in fact, in the discrete-time case, the exponential e ^ (j omega_0 n), that I've indicated here, may or may not be periodic depending on what the value of omega_0 is.

OK. Now, to summarize, in this lecture I've introduced and discussed a number of important basic signals. In particular, sinusoids and real and complex exponentials. One of the important outcomes of the discussion, emphasized further in the text, is that there are some very important similarities between them. But there are also some very important differences. And these differences will surface when we exploit sinusoids and complex exponentials as basic building blocks for more general continuous-time and discrete-time signals.

In the next lecture, what I'll discuss are some other very important building blocks, namely, what are referred to as step signals and impulse signals. And those, together with the sinusoidal signals and exponentials as we've talked about today, will really form the cornerstone for, essentially, all of the signal and system analysis that we'll be dealing with for the remainder of course. Thank you.