23

Mapping Continuous-Time Filters to Discrete-Time Filters

In Lecture 22 we introduced the z-transform. In this lecture we discuss some of the properties of the z-transform and show how, as a result of these properties, the z-transform can be used to analyze systems described by linear constant-coefficient difference equations. Toward this end, the three most significant properties are the linearity property, the time-shifting property, and the convolution property. As a consequence of the convolution property, the z-transform of the output of an LTI system is the product of the z-transform of the input and the z-transform of the system impulse response, referred to as the *system function*. The system function, for any specific value of z, say z_0, corresponds also to the change in (complex) gain of the eigenfunction z_0^n as it passes through the system. That is, $H(z)$ represents the spectrum of eigenvalues for discrete-time LTI systems, just as $H(s)$ represents the spectrum of eigenvalues for continuous time LTI systems.

Based on the linearity, time-shifting, and convolution properties, applying the z-transform to a linear constant-coefficient difference equation converts it to an algebraic equation that can be solved for the system function. Again, closely paralleling the discussion for continuous time, this specifies the algebraic expression for the system function but does not explicitly specify the associated ROC. However, if in addition the system is specified to be causal, then the ROC must lie outside the circle bounded by the outermost pole. Alternatively, if the system is known to be stable, the ROC of the system function must include the unit circle. It then extends inward and outward, in both cases until it reaches a pole (or the origin and/or infinity). Typically (but not always) in discussing systems described by linear constant-coefficient difference equations, we assume causality of the system. Just as with continuous-time systems, first- and second-order discrete-time difference equations play a particularly important role as building blocks for higher-order difference equations.

In designing a discrete-time system, a variety of design procedures is available for obtaining a linear constant-coefficient difference equation to meet or approximate a given set of system specifications. One particularly important class of such procedures corresponds to mapping continuous-time designs to discrete-time designs. This approach is motivated in part by the fact that continuous-time filter design has a long and rich history; to the extent that well-developed design procedures for continuous-time systems can be exploited in the design of discrete-time systems, they should be. Furthermore, in many applications discrete-time systems are used to process continuous-time signals by exploiting the concepts of sampling. In such cases, the discrete-time system to be designed and implemented is closely associated with a corresponding continuous-time system.

An often used but not highly desirable approach to mapping continuous-time systems to discrete-time systems is to replace derivatives in the differential equation describing the continuous-time system by simple forward or backward differences to obtain a discrete-time difference equation. The limitations of this approach are perhaps best understood by examining the corresponding mapping from the s-plane to the z-plane, from which it is evident that the frequency response can be severely distorted. In addition, with the use of forward differences, unstable discrete-time filters can result, even when the continuous-time filter from which it is derived is stable.

A second approach discussed is the impulse-invariant design procedure, whereby the discrete-time system function is determined in such a way that the impulse response of the discrete-time system corresponds to samples of the impulse response of the continuous-time system. This procedure can equivalently be interpreted as a mapping of the poles of the system function. In terms of the corresponding frequency responses, the discrete-time frequency response is identical in shape to the continuous-time frequency response except for possible distortion due to aliasing. Consequently, it is useful only for mapping continuous-time systems for which the frequency response is bandlimited.

Suggested Reading

Section 10.5, Properties of the z-Transform, pages 649–654

Section 10.7, Analysis and Characterization of LTI Systems Using z-Transforms, pages 655–658

Section 10.4, Geometric Evaluation of the Fourier Transform from the Pole-Zero Plot, pages 646–648

Section 10.8, Transformations Between Continuous-Time and Discrete-Time Systems, pages 658–665

MARKERBOARD 23.1 (a)

z-Transform

$X(z)=\sum_{n=-\infty}^{+\infty} x[n]z^{-n}$

$x[n]=\frac{1}{2\pi j}\oint X(z)z^{n-1}dz$

$X(z)\Big|_{z=e^{j\Omega}} = \mathcal{F}\{x[n]\}$

$z=re^{j\Omega}$

$X(z)=\mathcal{F}\{x[n]r^{-n}\}$

Converges for some values of r and not others ⇒ ROC

first-order difference equation

$y[n]-ay[n-1]=x[n]$

⇕ ⇕ ⇕

$Y(z)-az^{-1}Y(z)=X(z)$

$Y(z)=\underbrace{\frac{1}{1-az^{-1}}}_{H(z)}X(z)$

Causality $|z|>|a|$

$h[n]=a^n u[n]$

Second-order difference equation

$y[n]+2r\cos\theta\, y[n-1]+r^2y[n-2]=x[n]$

$Y(z)[1+2r\cos\theta\, z^{-1}+r^2z^{-2}]=X(z)$

$Y(z)=\underbrace{\frac{1}{1+2r\cos\theta\, z^{-1}+r^2z^{-2}}}_{H(z)}X(z)$

$\cos\theta<1$ ⇒ complex poles

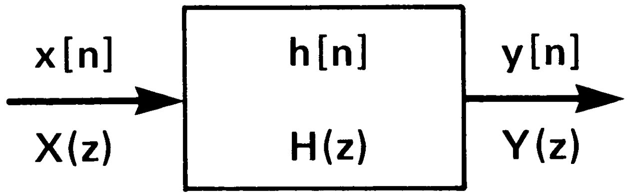
poles at $re^{\pm j\theta}$

z-TRANSFORM PROPERTIES

SIGNAL	TRANSFORM	ROC
$ax_1[n]+bx_2[n]$	$aX_1(z)+bX_2(z)$	at least $R_1\cap R_2$
$x[n-n_0]$	$z^{-n_0}X(z)$	R
$x_1[n]*x_2[n]$	$X_1(z)X_2(z)$	at least $R_1\cap R_2$

TRANSPARENCY 23.1
Some properties of the z-transform.

TRANSPARENCY 23.2
Equivalence of the constraints on a system impulse response for stability and for the existence of the Fourier transform.



$$y[n] = h[n] * x[n]$$

$$Y(z) = H(z)\,X(z)$$

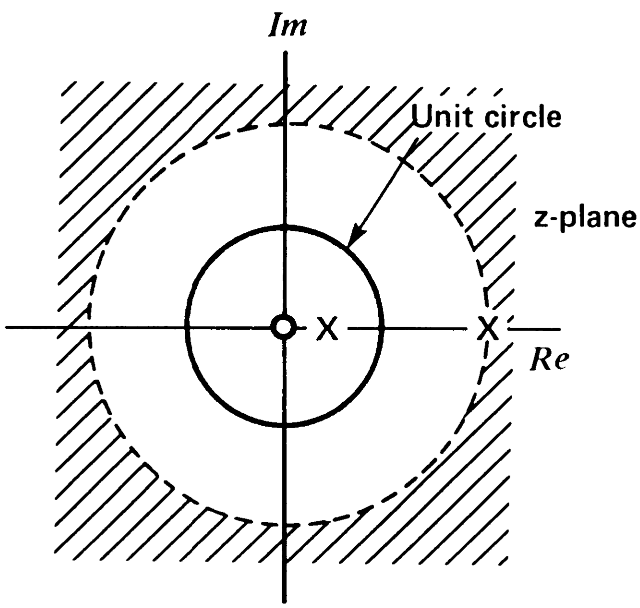
$$\text{stable} \iff \sum_{n=-\infty}^{+\infty} |h[n]| < \infty$$

$$\mathfrak{F}\{h[n]\} \iff \sum_{n=-\infty}^{+\infty} |h[n]| < \infty$$

TRANSPARENCY 23.3
Relationship between the properties of stability and causality and corresponding constraints on the ROC of the system function.

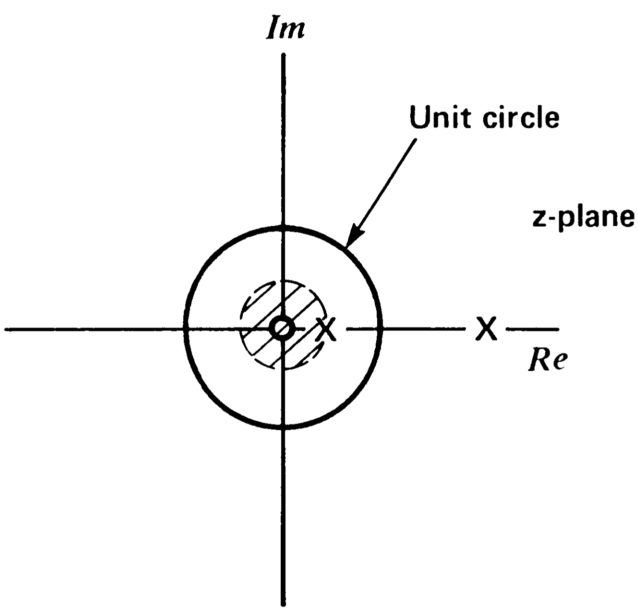
- stable <=> ROC of H(z) includes unit circle in z-plane
- causal => h[n] right-sided
  => ROC of H(z) outside outermost pole
- causal and stable <=> All poles inside unit circle

$$X(z) = \frac{z}{(z-\frac{1}{3})(z-2)}$$



TRANSPARENCY 23.4
Transparencies 23.4–23.6 show a specified system function and the relationship between the three choices for the ROC and the properties of system stability and causality. Here, the system is causal and unstable.

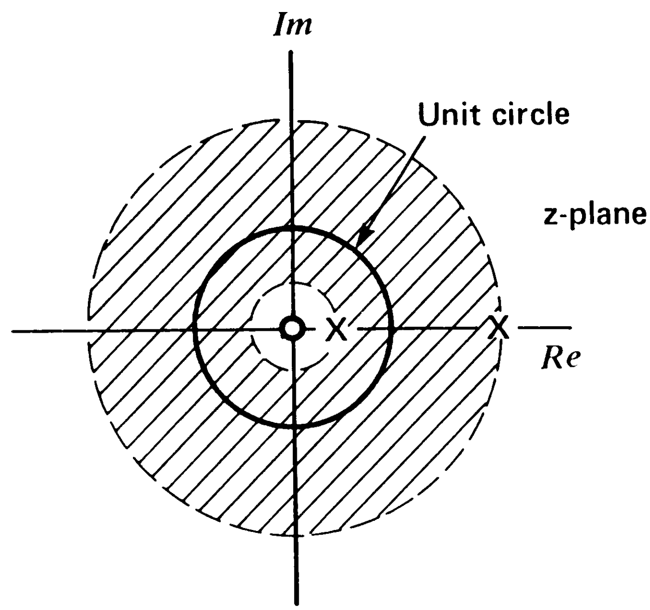
$$X(z) = \frac{z}{(z-\frac{1}{3})(z-2)}$$



TRANSPARENCY 23.5
The system is unstable and not causal.

TRANSPARENCY 23.6
The system is stable and not causal.

$$X(z) = \frac{z}{(z - \frac{1}{3})(z - 2)}$$



TRANSPARENCY 23.7
Some properties of the z-transform. [Transparency 23.1 repeated]

z-TRANSFORM PROPERTIES

SIGNAL	TRANSFORM	ROC
$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	at least $R_1 \cap R_2$
$x[n-n_o]$	$z^{-n_o} X(z)$	R
$x_1[n] * x_2[n]$	$X_1(z) X_2(z)$	at least $R_1 \cap R_2$

MARKERBOARD
23.1 (b)

z-Transform

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

$$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$$

$$X(z)\Big|_{z=e^{j\Omega}} = \mathcal{F}\{x[n]\}$$

$$z = re^{j\Omega}$$

$$X(z) = \mathcal{F}\{x[n]r^{-n}\}$$

Converges for some values of r and not others ⇒ ROC

first-order difference equation

$$y[n] - ay[n-1] = x[n]$$

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$$Y(z) - az^{-1}Y(z) = X(z)$$

$$Y(z) = \underbrace{\frac{1}{1-az^{-1}}}_{H(z)} X(z)$$

Causality $|z| > |a|$

$$h[n] = a^n u[n]$$

Second-order difference equation

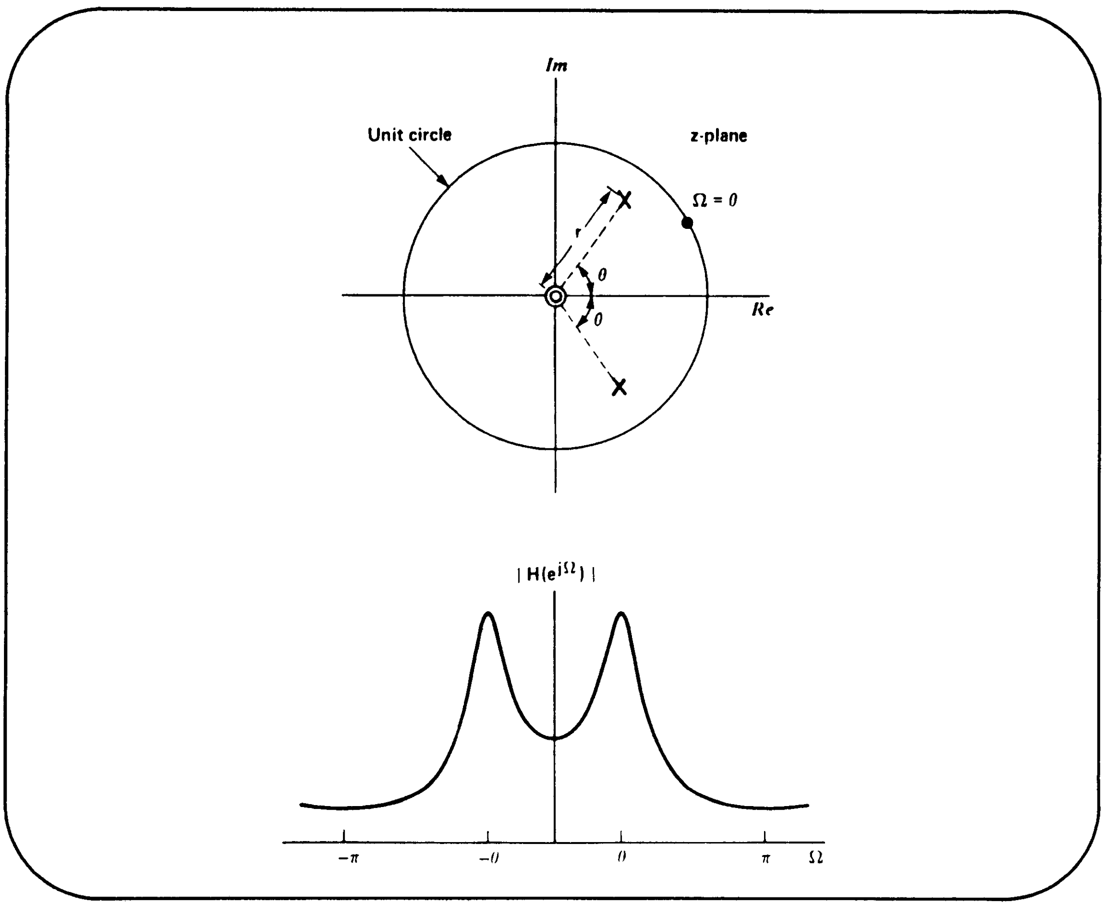
$$y[n] + 2r\cos\theta\, y[n-1] + r^2 y[n-2] = x[n]$$

$$Y(z)[1 + 2r\cos\theta\, z^{-1} + r^2 z^{-2}] = X(z)$$

$$Y(z) = \underbrace{\frac{1}{1+2r\cos\theta\, z^{-1} + r^2 z^{-2}}}_{H(z)} X(z)$$

$\cos\theta < 1 \Rightarrow$ complex poles

poles at $re^{\pm j\theta}$



TRANSPARENCY
23.8
Frequency response for an underdamped second-order system.

MARKERBOARD
23.2

Mapping of Continuous-Time Filters to Discrete-Time Filters

- D-T Processing of C-T Signals
- Exploit established design procedures for C-T filters

$H_c(s)$

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

$H_d(z)$

$$\sum_{k=0}^{N} c_k y[n-k] = \sum_{k=0}^{M} d_k x[n-k]$$

C-T		D-T
$H_c(s)$	$\longrightarrow$	$H_d(z)$
$h_c(t)$	$\longrightarrow$	$h_d[n]$
$j\omega$-axis	$\longrightarrow$	unit circle
stable	$\longrightarrow$	stable

Backward Difference

$$\frac{dy(t)}{dt} \longrightarrow \frac{y[n]-y[n-1]}{T}$$

$$sY(s) \longrightarrow \frac{1-z^{-1}}{T} Y(z)$$

$$H_d(z) = H_c(s)\Big|_{s=\frac{1-z^{-1}}{T}}$$

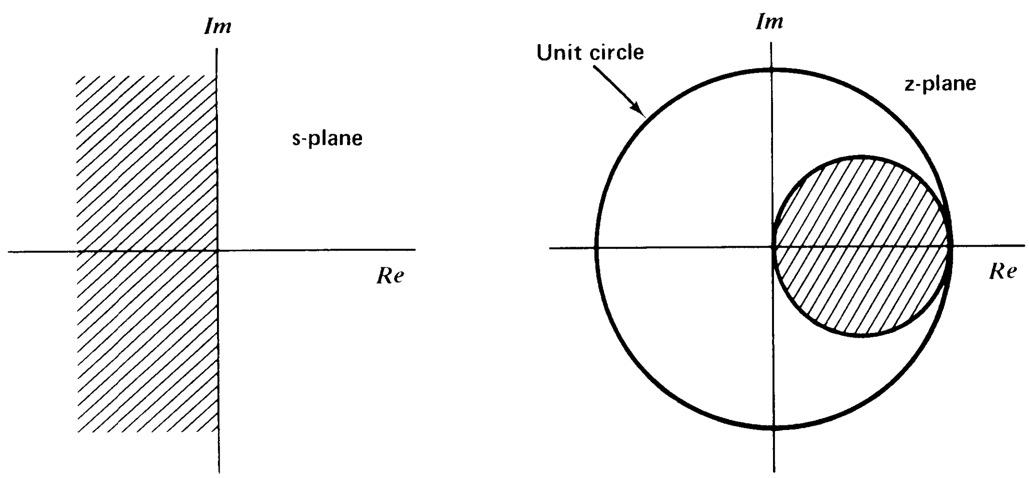
Forward Difference

$$\frac{dy(t)}{dt} \longrightarrow \frac{y[n+1]-y[n]}{T}$$

$$sY(s) \longrightarrow \frac{z-1}{T} Y(z)$$

$$H_d(z) = H_c(s)\Big|_{s=\frac{z-1}{T}}$$

TRANSPARENCY
23.9
Mapping from the s-plane to the z-plane that results when a differential equation is mapped to a difference equation by replacing derivatives with differences.



MARKERBOARD 23.3

Impulse Invariance

$$h_d[n] = h_c(nT)$$

$$H_d(e^{j\Omega}) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} H_c\left[j\left(\frac{\Omega}{T} - \frac{2\pi k}{T}\right)\right]$$

$$H_c(s) = \sum_{k=1}^{N} \frac{A_k}{s - s_k}$$

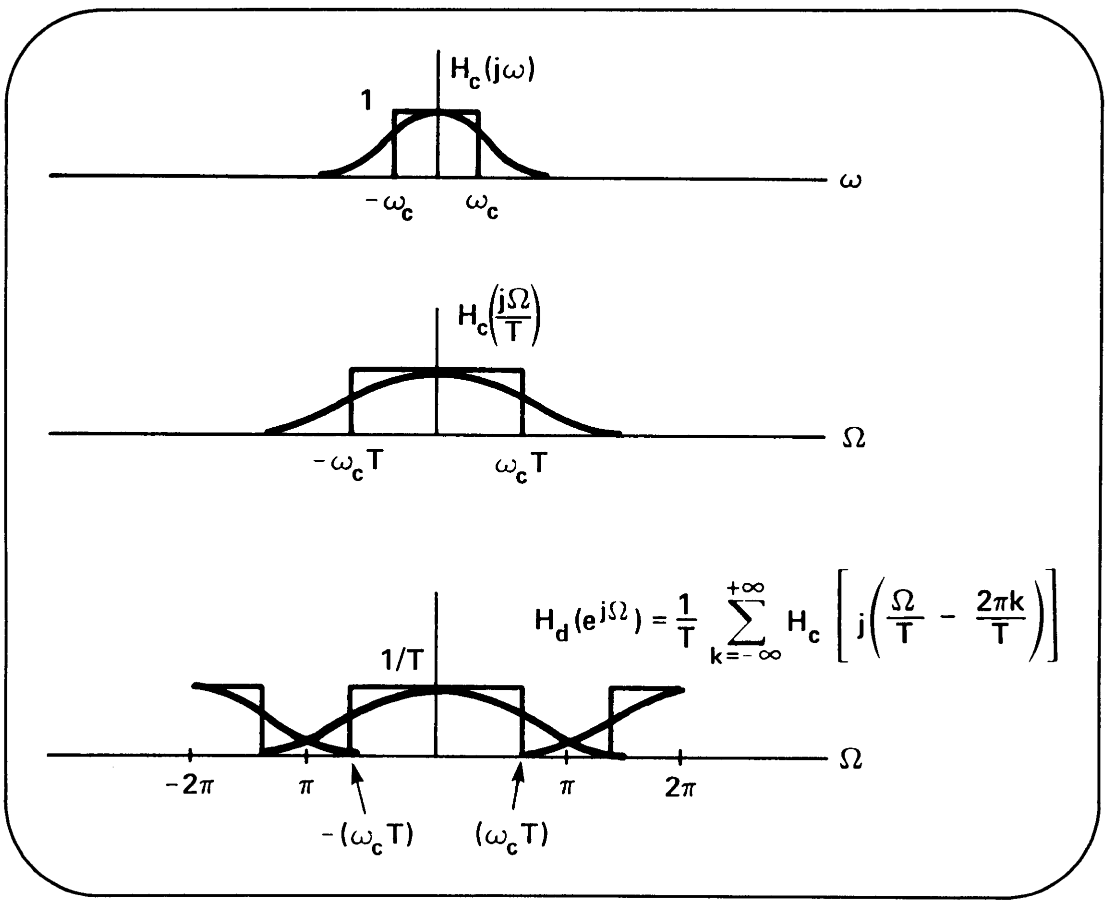
$$h_c(t) = \sum_{k=1}^{N} A_k e^{s_k t} u(t)$$

$$h_d[n] = \sum_{k=1}^{N} A_k e^{s_k nT} u[n]$$

$$h_d[n] = \sum_{k=1}^{N} A_k (e^{s_k T})^n u[n]$$

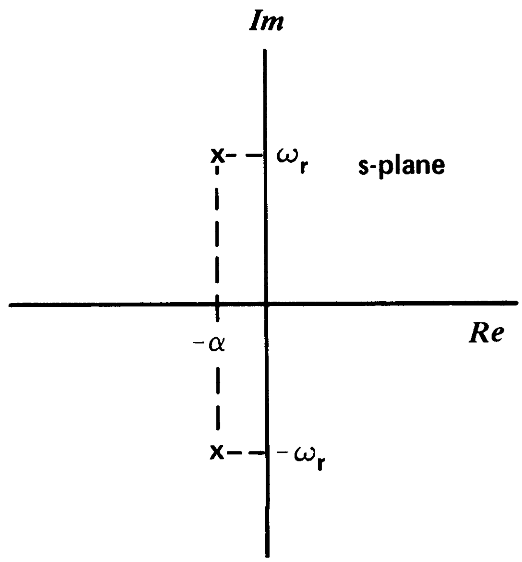
$$H_d(z) = \sum_{k=1}^{N} \frac{A_k}{1 - e^{s_k T} z^{-1}}$$

- pole at $s = s_k$ $\Rightarrow$ pole at $z = e^{s_k T}$
- Coefficients A_k preserved



TRANSPARENCY 23.10
Illustration of spectra associated with impulse invariance.

TRANSPARENCY 23.11
Continuous-time second-order transfer function mapped to a discrete-time system function using impulse invariance.

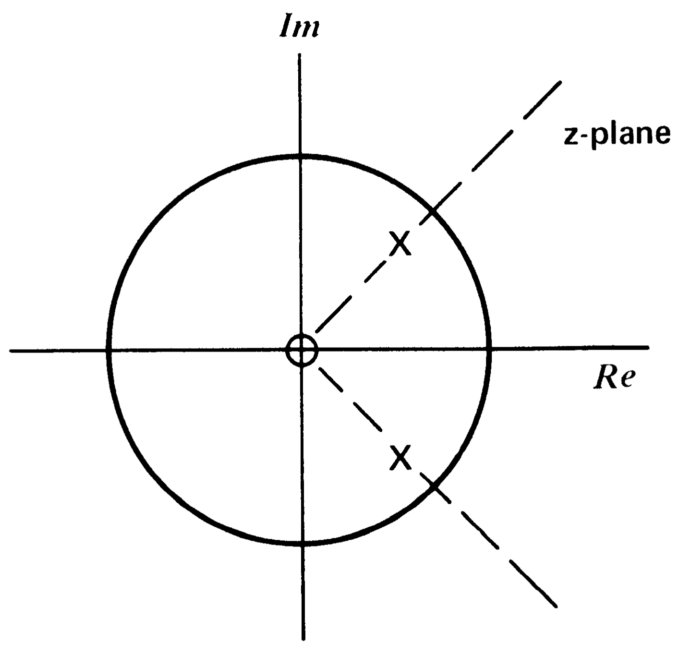


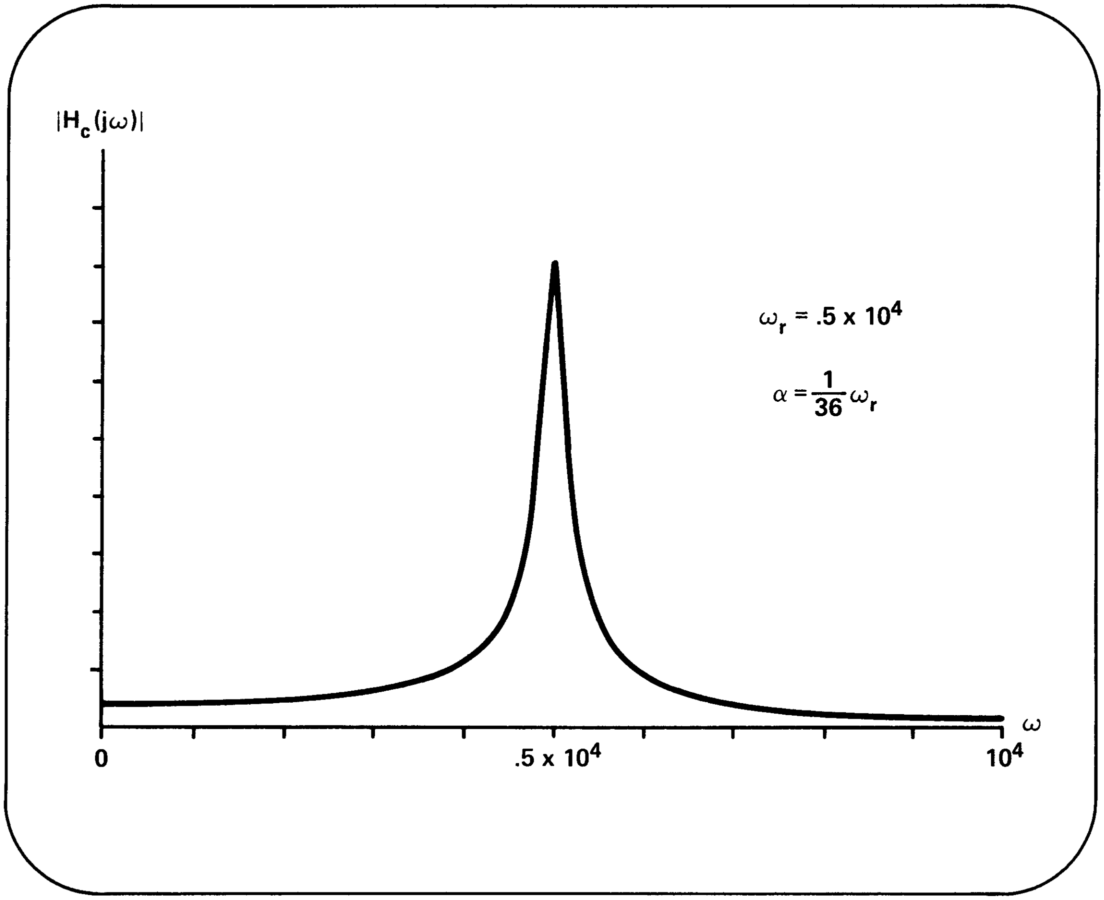
$$H_c(s) = \frac{2\omega_r}{(s+\alpha+j\omega_r)\,(s+\alpha-j\omega_r)}$$

$$= \frac{j}{(s+\alpha+j\omega_r)} + \frac{-j}{(s+\alpha-j\omega_r)}$$

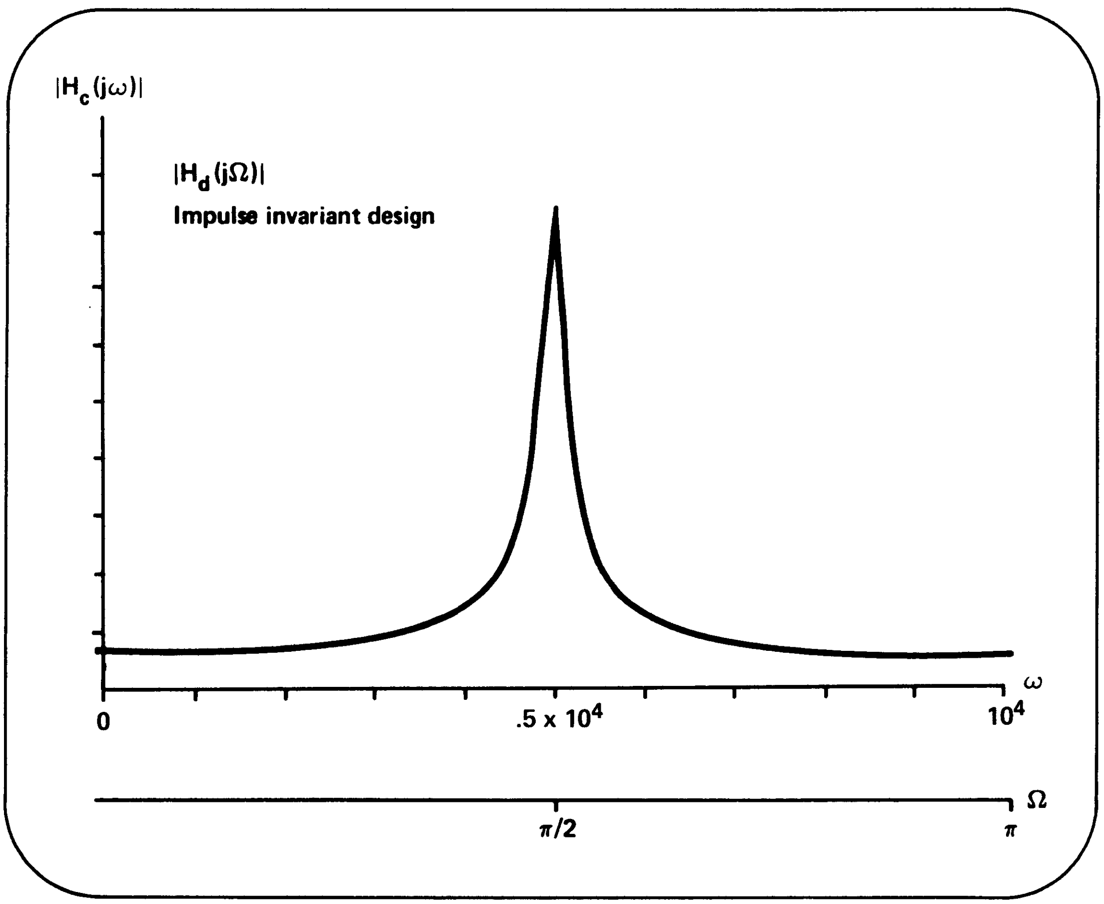
TRANSPARENCY 23.12
Discrete-time system function and pole-zero plot resulting from impulse invariance applied to the system function in Transparency 23.9.

$$H_d(z) = \frac{j}{1-e^{-\alpha T}\,e^{-j\omega_r T}\,z^{-1}} + \frac{-j}{1-e^{-\alpha T}\,e^{j\omega_r T}\,z^{-1}}$$





TRANSPARENCY 23.13
Transparencies 23.13–23.15 show a comparison of the frequency response when a second-order continuous-time system is mapped to a second order discrete-time system using impulse invariance and backward differences. Shown here is a continuous-time frequency response.



TRANSPARENCY 23.14
Discrete-time frequency response resulting from impulse invariance.

TRANSPARENCY 23.15
Discrete-time frequency response resulting from the use of backward differences.