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Continuum Electromechanics

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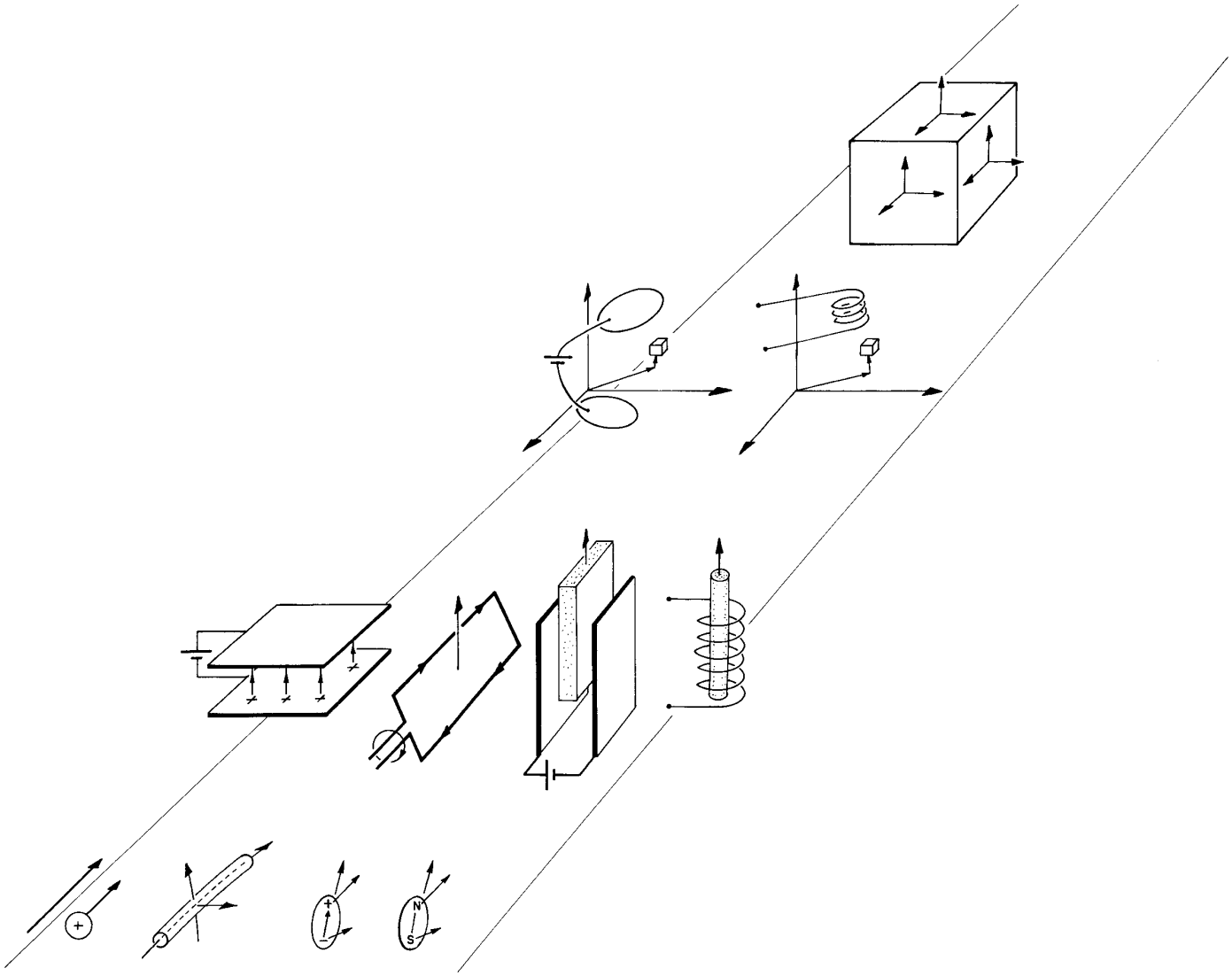
3

Electromagnetic Forces, Force Densities and Stress Tensors



3.1 Macroscopic versus Microscopic Forces

Most important in this chapter is the distinction between forces on fundamental particles and forces on macroscopic media. It is common to speak of the "force on a charge" or the "force on a current" even though what is meant is the force on ponderable material. Interest might actually be in electric and magnetic forces acting on collections of fundamental charge carriers. (Motions of electron beams in vacuum are an example. The charged particles in that case constitute the continuum, in the sense that it is the electron inertia that enters into the equation of motion.) But, more commonly, the charged particles are imbedded in media, and it is the resulting force on the material that is of interest. Examples are as obvious as the electrical force of attraction between the capacitor plates of an electrostatic voltmeter or the magnetic torque exerted on current-carrying conductors in a meter movement.

Section 3.2 develops a specific model to illustrate how momentum imparted to charged particles by the fields is transferred to the neutral media that support those particles. That macroscopic forces are more than simply an average over the forces on fundamental charges is further emphasized by considering the practical cases of polarization and magnetization forces. Force densities of engineering significance exist even in regions where the free charge and free current (and for that matter polarization charge or magnetization charge) are absent. Such forces can be associated with a microscopic picture, discussed in Sec. 3.6, in which electrical forces on dipoles are transferred to the media.

Although the dipole model is useful for forming a microscopic picture of electric polarization forces, it is restricted to cases where the dipoles do not significantly interact. In the pursuit of a less restricted force density, developments in Secs. 3.7-3.8 are based on such measured macroscopic parameters as the permittivity and permeability. It is the business of thermodynamics to convert that information into the desired force densities. In its own way, the line of reasoning presented in Secs. 3.5, 3.7 and 3.8 exemplifies a more basic point of view than one geared to a particular microscopic model. Thermodynamic concepts provide a means for replacing detailed and specialized derivations by carefully defined physical measurements.

The stress-tensor representation of electromagnetic forces which concludes this chapter will see continual application in the following chapters. The tensor concept itself, introduced in Sec. 3.9, will also be applied to the formulation of continuum mechanical and electromechanical equations.

3.2 The Lorentz Force Density

Although macroscopic forces were the first measured in the development of electricity and magnetism, it is now normally accepted that the fundamental force is that on a "test" charge. This charge might be a single electron in free space. If the charged particle has a total charge q and moves with a velocity $\vec{v}_p$, then the Lorentz force acting on the particle supporting the charge is

$$\vec{f} = q\vec{E} + q\vec{v}_p \times \mu_o\vec{H} \tag{1}$$

This statement, like the electrodynamic laws summarized in Chap. 2, is an empirical one. In most of the areas of continuum electromechanics, it is forces due to many charges that are of interest, and it is therefore appropriate to sum the individual forces of Eq. 1 over the charges within a given unit of volume to arrive at the Lorentz force density

$$\vec{F} = \rho_f\vec{E} + \vec{J}_f \times \mu_o\vec{H} \tag{2}$$

Incremental volumes of interest have dimensions much greater than the characteristic distances between particles. But also, for the average electrical field to have meaning, it must be primarily due to sources external to the differential volume of interest. This ensures that, over an incremental volume, each particle experiences essentially the same electric field. The contribution to the field of the charges within the differential volume is negligible. Similar arguments apply to the magnetic field intensity, which must be produced over a given differential volume largely by currents outside the volume.

Equation 2 represents the force density acting on a ponderable medium if means are available for the force on the particles to be transmitted to the medium. The mechanisms by which this happens are diverse, and implicit to the conduction process. Whether the fundamental carriers are electrons in a metal, holes and electrons in a semiconductor or ions in a liquid or gas, the average motions of fundamental charge carriers are superimposed on random motions. The flights of fundamental carriers are interrupted by collisions with lattice molecules (in a solid) or molecules that are themselves in a Brownian equilibrium (in a liquid or gas) with a frequency that is usually extremely high compared to reciprocal times of interest. These collisions transfer momentum from the fundamental charge carriers to the ponderable medium.

To more fully appreciate the transition from the force acting on fundamental carriers, Eq. 1, to that on a material, Eq. 2, it is helpful to make a formal derivation. Although the discussion leads to rather general conclusions, only two families of carriers are now considered, one positive with charge per particle q_+ and number density n_+ and the other negative with a magnitude of charge q_- and number density n_-. The average Lorentz force, Eq. 1, is in equilibrium with an average force representing the effect of collisions on the net migration of the particles:

$$\begin{aligned} q_+\vec{E} + q_+(\vec{v}_+ + \vec{v}) \times \mu_o\vec{H} &= m_+\nu_+\vec{v}_+ \\ -q_-\vec{E} - q_-(\vec{v}_- + \vec{v}) \times \mu_o\vec{H} &= m_-\nu_-\vec{v}_- \end{aligned} \tag{3}$$

The retarding forces on the right are much as would be conceived for a swarm of macroscopic particles moving through a viscous liquid. The average carrier velocities $\vec{v}_\pm$ are measured relative to the medium, which itself has the velocity $\vec{v}$. Hence, on the right it is relative velocities of particles and medium that appear, while in the Lorentz force it is total particle velocities that are appropriate. The coefficients for the collisional forces are written as the product of the particle masses $m_\pm$ and collision frequencies $\nu_\pm$ as a matter of convention. Note that the inertial force on the carriers is ignored compared to that due to collisions. This approximation would be invalidated in a plasma if the frequency of an applied electric field intensity were extremely high. But, in many conductors and certainly in the most usual electromechanical situations, the inertial effects of the charge carriers can be ignored (see (Problem 3.3.1.).

The charge density and current density are written in terms of the microscopic variables as

$$\rho_f = n_+q_+ - n_-q_- \tag{4}$$

$$\begin{aligned} \vec{J}_f &= n_+q_+(\vec{v}_+ + \vec{v}) - n_-q_-(\vec{v}_- + \vec{v}) \\ &= n_+q_+\vec{v}_+ - n_-q_-\vec{v}_- + \rho_f\vec{v} \end{aligned} \tag{5}$$

The average force density acting on the ponderable medium is the sum of the right-hand sides of Eq. 3, respectively, multiplied by the particle densities $n_\pm$:

$$\vec{F} = n_+m_+\nu_+\vec{v}_+ + n_-m_-\nu_-\vec{v}_- \tag{6}$$

The point in writing this equation is to formalize the statement that, through some collisional process, the force on the fundamental carriers becomes the force on the medium. It is evident from the next step that, at least in so far as the Lorentz force density is concerned, the details of the collisional equilibrium are not important. The left-hand sides of Eq. 3 (regardless, for example, of whether $m_\pm\nu_\pm$ are functions of $v_\pm$ or are constant) are substituted for the respective terms in Eq. 6 to obtain

$$\vec{F} = (n_+q_+ - n_-q_-)\vec{E} + [(n_+q_+\vec{v}_+ - n_-q_-\vec{v}_-) + (n_+q_+ - n_-q_-)\vec{v}] \times \mu_o\vec{H} \tag{7}$$

In view of the definitions given by Eqs. 4 and 5, this expression is the Lorentz force density of Eq. 2. Its validity hinges on there being an instantaneous equilibrium between the forces on the fundamental carriers and the "collisions" with the ponderable medium, but not on the details of that interaction.

3.3 Conduction

There are three objectives in this section. The first is to have a microscopic picture of the carrier motions to associate with ohmic or unipolar conduction models. The second is to illustrate how constitutive laws for media in motion can be derived from models based on particular microscopic models, or (on the basis of the field transformations) found by generalizing empirically determined laws established in the laboratory for materials at rest. Finally, a byproduct of the discussion is an introduction to Hall effect.

Consider the carrier motions represented by Eqs. 3.2.3, with the magnetic field $\vec{H} = H_o\vec{i}_x$ externally imposed. The components of these equations then respectively become

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \mp b_\pm\mu_oH_o \\ 0 & \pm b_\pm\mu_oH_o & 1 \end{bmatrix} \begin{bmatrix} v_{x\pm} \\ v_{y\pm} \\ v_{z\pm} \end{bmatrix} = \begin{bmatrix} \pm b_\pm E_x \\ \pm b_\pm E_y \pm b_\pm v_z\mu_oH_o \\ \pm b_\pm E_z \mp b_\pm v_y\mu_oH_o \end{bmatrix} \tag{1}$$

where particle mobilities are defined as $b_{\pm} = q_{\pm}/m_{\pm}\nu_{\pm}$.

These three equations can be inverted to find the relative carrier velocities in terms of $(\vec{E},\vec{H},\vec{v})$:

$$\begin{bmatrix} v_{x\pm} \\ v_{y\pm} \\ v_{z\pm} \end{bmatrix} = \frac{1}{\Delta_{\pm}} \begin{bmatrix} \frac{\pm b_{\pm}}{\Delta_{\pm}} & 0 & 0 \\ 0 & \pm b_{\pm} & b_{\pm}^2 \mu_o H_o \\ 0 & -b_{\pm}^2 \mu_o H_o & \pm b_{\pm} \end{bmatrix} \begin{bmatrix} E_x \\ E_y + v_z \mu_o H_o \\ E_z - v_y \mu_o H_o \end{bmatrix} \tag{2}$$

where $\Delta_{\pm} = 1 + .(\mu_o H_o b_{\pm})^2$.

These velocity components can now be introduced into Eq. 3.2.5 to express the free current density as

$$\vec{J}_f = (n_+q_+b_+ + n_-q_-b_-)\vec{E}'_x\vec{i}_x + \left(\frac{n_+q_+b_+}{\Delta_+} + \frac{n_-q_-b_-}{\Delta_-}\right)(E'_y\vec{i}_y + E'_z\vec{i}_z) + \left(\frac{n_+q_+b_+^2}{\Delta_+} - \frac{n_-q_-b_-^2}{\Delta_-}\right)\mu_o\vec{E}' \times \vec{H}_o + \rho_f\vec{v} \tag{3}$$

where $\vec{E}' \equiv \vec{E} + \vec{v} \times \mu_o\vec{H}$ is the electric field in a frame of reference moving with the material (for a magnetoquasistatic system).

From Eq. 3, it is clear that there are two components to the current density, one in the direction of the imposed electric field and the second perpendicular to it. The latter term is called the Hall current and is due to the tendency of the particles to move perpendicular to their own velocity and to the imposed magnetic field intensity. This last term is ignorable if

$$\mu_o H_o b_{\pm} \ll 1 \tag{4}$$

A typical magnetic flux density is $\mu_o H_o = 1$ (10,000 gauss, which is in the range where magnetic materials saturate). Electrons in copper have a mobility on the order of 3×10^{-3} m^2/volt sec, so that the parameter on the left is then much less than 1. Ions in liquids have mobilities that are typically 5×10^{-8} m^2/volt sec and the approximation is even better. But in silicon or germanium, where the electron mobility is in the range of 10^{-1} m^2/volt sec, the Hall effect is coming into play by the time $\mu_o H_o$ is of the order of unity. With the inequality of Eq. 4 satisfied, Eq. 3 reduces to the familiar form

$$\vec{J}_f = (n_+q_+b_+ + n_-q_-b_-)\vec{E}' + \rho_f\vec{v} \tag{5}$$

If the number density of charge carriers n_+ and/or n_- remains essentially the same in spite of the application of $\vec{E}$, then the factor multiplying $\vec{E}$ in Eq. 5 is usefully regarded as a parameter characterizing the material, the electrical conductivity σ. This case of ohmic conduction is displayed by materials ranging from metallic conductors, where the carriers are electrons and essentially immobile ions, to electrolytes, where ions of at least two species participate in the conduction. In any of these cases, for the ohmic model to be valid, the conduction must involve at least two species with both n_+q_+ and n_-q_- greatly exceeding the net charge ρ_f. By introducing the conductivity as a parameter, the detailed analysis necessary to determine the self-consistent distributions of the individual carriers is avoided. But to examine the conditions under which the conductivity model is valid, it is necessary to formulate the laws that govern the self-consistent carrier motions. This is best done in the context of molecular diffusion (Chap. 10) so that other important limitations on the model can also be identified.

Even though in accounting for conduction it is useful to have in mind microscopic mechanisms, it is also important to recognize the far-reaching implications of empirical relations. Given any conduction law based on laboratory measurements made with a fixed sample, effects of material motion can be brought in by using the transformation laws. For example, if it is known that the conductor obeys Ohm's law when stationary, then in a primed inertial frame moving with the velocity $\vec{v}$ of the conductor, the experiment shows that

$$\vec{J}'_f = \sigma\vec{E}' \tag{6}$$

In an electroquasistatic system, including polarization, $\vec{J}'_f = \vec{J}_f - \rho_f\vec{v}$ (Eq. 2.5.12a) and $\vec{E}' = \vec{E}$ (Eq. 2.5.9a). Hence, Eq. 6 becomes Eq. 5. In a magnetoquasistatic system, including magnetization,

$\vec{J}_f' = \vec{J}_f$ (Eq. 2.5.11b) and $\vec{E}' = \vec{E} + \vec{v} \times \mu_o \vec{H}$ (Eq. 2.5.12b). Substitution in Eq. 6 now gives Eq. 5, except for the charge convection term $\rho_f \vec{v}$. In a magnetoquasistatic system, this term is second-order, as will be argued in the next section.

Fundamental to the use of an empirical law determined for the stationary material is the assumption that material acceleration and deformation do not influence the conduction. In any case, if acceleration did effect the conduction, the close tie between conduction and the Lorentz force density, illustrated in this and the previous section, calls into question the notion that the electromechanics can be modeled by a single continuum subject to the Lorentz force density.

3.4 Quasistatic Force Density

The Lorentz force density, Eq. 3.2.2, is composed of what will be termed, respectively, an electric force density and a magnetic force density

$$\vec{F} = \rho_f \vec{E} + \vec{J}_f \times \mu_o \vec{H} \tag{1}$$

It is found in a wide range of applications that the force density is predominantly one or the other of these contributions. Polarization and magnetization force densities, not included in Eq. 1, are similarly identified with the respective quasistatic systems. In this section, dimensional arguments are given that demonstrate that the electric force density generally dominates in electroquasistatic systems, while the magnetic force density dominates in magnetoquasistatic systems.

The line of reasoning is an extension of that introduced in Sec. 2.2. The force density is normalized in accordance with Eq. 2.3.4 and the free current density is represented as having the form of Eq. 2.3.1. Thus,

$$\vec{F} = \frac{\varepsilon_o \mathscr{E}^2}{\ell} \left[\rho_f \vec{E} + \frac{\tau_m}{\tau} \left(\sigma \vec{E} + \frac{\tau_e}{\tau} \vec{J}_v \right) \times \vec{H} \right] \qquad \text{EQS} \tag{2}$$

$$\vec{F} = \frac{\mu_o \mathscr{H}^2}{\ell} \left[\left(\frac{\tau_{em}}{\tau} \right)^2 \rho_f \vec{E} + \left(\frac{\tau_m}{\tau} \vec{E} + \vec{J}_v \right) \times \vec{H} \right] \qquad \text{MQS} \tag{3}$$

The relative values of the time constants are summarized by Fig. 2.3.1. In the electroquasistatic system, $\tau_m/\tau << 1$ and $\tau_m \tau_e/\tau^2 = (\tau_{em}/\tau)^2 << 1$. Hence, the free charge density term is zero-order in Eq. 1, and the magnetic term is consistently ignored.[1] In the magnetoquasistatic force density of Eq. 3, $(\tau_{em}/\tau)^2 << 1$, and the free charge force density is negligible compared to the magnetic term. Hence, the second term of Eq. 1 is used to the exclusion of the first in magnetoquasistatic systems.

3.5 Thermodynamics of Discrete Electromechanical Coupling

In this section, the thermodynamic electric and magnetic energy storage subsystems are expanded to include the possibility of a finite number of discrete mechanical displacements of macroscopic material. Based on the notion of an energy function and a thermodynamic equilibrium, the force of electrical origin associated with each of these displacements is determined. Typically, the method exploits a knowledge of the electrical terminal relations to determine the forces. The approach is generalized in Secs. 3.7 and 3.8, where constitutive laws are the basis for finding the force density of electric origin. Except for mathematical manipulations, the derivations now reviewed draw upon all of the demanding issues confronted later in deriving force densities.

Electroquasistatic Coupling: An example of a lumped-parameter electroquasistatic system is given with Fig. 2.11.1, including a schematic representation of a finite number of mechanical displacements. Associated with each of the displacements is an electromechanical force tending to displace a lumped element by an amount $\delta\xi_i$.

Conservation of energy for the system with the geometry fixed is expressed by Eq. 2.13.8. Now, an incremental increase in the total energy caused by placing an increment of charge δq_i on an electrode having the voltage v_i can be diminished by an amount equal to the work done on the external environment by the forces of electrical origin acting through the displacements of the associated mechanical entities. Thus, energy conservation requires that

$$\delta w = \sum_{i=1}^{n} v_i \delta q_i - \sum_{j=1}^{m} f_j \delta \xi_j; \quad w = w(q_1 \cdots q_n, \xi_1 \cdots \xi_m) \tag{1}$$

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1. Electrons in vacuum can have a velocity approaching that of light. In that case an imposed magnetic field can have a crucial effect on the EQS dynamics (See Sec. 11.2).

Given the charges $q_1 \cdots q_n$ and the displacement $\xi_1 \cdots \xi_m$ as independent variables, the energy function is uniquely determined. The "displacements" should be recognized as generalized variables in that they could just as well be angular deflections, in which case the associated "forces" would be torques.

To determine w, constitutive relations $v_i(q_1 \cdots q_n, \xi_1 \cdots \xi_n)$ must be known so that Eq. 1 can be integrated. The integration is a line integral in a state-space composed of the independent variables. Because the f_j's are not known, and are defined as equal to zero in the absence of electrical excitations, integration on the mechanical variables ξ_j is carried out first. This gives no contribution because as the displacements are brought to their final values, $f_j = 0$ (no work is required to assemble the system with the q_j's = 0). Then, the integration on successive electrical variables is carried out, first on q_1 with all other q_j's = 0, then on q_2 with q_1 at its final value and all others zero, etc. Formally, the integration of Eq. 1 gives

$$w = \sum_{j=1}^{n} \int_{o}^{q_j} v_j(q_1 \cdots q_j', 0 \cdots 0, \xi_1, \xi_2 \cdots \xi_m) \delta q_j' \tag{2}$$

Because the energy function is a state function specified by the independent variables, an incremental change in the total energy can also be written as

$$\delta w = \sum_{i=1}^{n} \frac{\partial w}{\partial q_i} \delta q_i + \sum_{j=1}^{m} \frac{\partial w}{\partial \xi_j} \delta \xi_j \tag{3}$$

If the q's and the ξ's are independent variables in the sense that Eqs. 1 and 3 hold for <u>arbitrary</u> combinations of incremental changes in these electrical and mechanical variables, then

$$v_i = \frac{\partial w}{\partial q_i} ; \quad f_j = - \frac{\partial w}{\partial \xi_j} \tag{4}$$

Note that the q's and ξ's are not necessarily independent of each other unless the system is isolated from the total system in which it is imbedded. Given w from Eq. 2, the electrical forces are determined.

A consequence of the conservation of energy expressed by Eq. 1 is the reciprocity condition between pairs of terminal variables. For example, derivatives of Eq. 4a, first with respect to q_j and then of the same equation but with i replaced by j, and with respect to q_i, are related by

$$\frac{\partial v_i}{\partial q_j} = \frac{\partial^2 w}{\partial q_i \partial q_j} = \frac{\partial v_j}{\partial q_i} \tag{5}$$

Other reciprocity conditions follow from Eq. 4 by taking cross-derivatives to relate forces and voltages to each other.

In dealing with practical lumped-parameter systems, it is often convenient to use the voltages rather than the charges as independent variables. If all of the voltages are to be independent variables, it is appropriate to recognize that

$$\sum_{i=1}^{n} v_i \delta q_i = \sum_{i=1}^{n} [\delta(v_i q_i) - q_i \delta v_i] \tag{6}$$

so that substitution into Eq. 1 gives

$$\delta w' = \sum_{i=1}^{n} q_i \delta v_i + \sum_{j=1}^{m} f_j \delta \xi_j \tag{7}$$

where a coenergy function has been defined in terms of the energy function as

$$w'(v_1 \cdots v_n, \xi_1 \cdots \xi_m) \equiv \sum_{i=1}^{n} v_i q_i - w \tag{8}$$

The coenergy function is a particular case of an arbitrarily large number of functions that can be defined. Any combination of charges and voltages can be independent variables, and a hybrid energy function, appropriately defined as a state function of this combination. With the voltages as independent variables, an equation similar to Eq. 2 is found with the charges replaced by the voltages, and the voltages and displacements the independent variables:

$$q_i = \frac{\partial w'}{\partial v_i} ; \quad f_j = \frac{\partial w'}{\partial \xi_j} \tag{9}$$

The coenergy function, like the energy function, is found from purely electrical considerations, as described in Sec. 2.13.

Magnetoquasistatic Coupling: Lumped-parameter electromechanical coupling in a magnetic field system, described schematically by Fig. 2.12.1, can be given the same thermodynamic representation as that outlined for electroquasistatic systems. The statement of conservation of energy for the system of discrete coils and mechanical displacements is the generalization of Eq. 2.14.11, with the addition of the mechanical work done as an electrical force f_j causes an incremental displacement $\delta\xi_j$:

$$\delta w = \sum_{i=1}^{n} i_i \delta\lambda_i - \sum_{j=1}^{m} f_j \delta\xi_j \tag{10}$$

All of the arguments given for the electric systems follow for the magnetic field systems if variables are identified:

$$q_i \rightarrow \lambda_i, \; v_i \rightarrow i_i \tag{11}$$

$$w = w(\lambda_1 \cdots \lambda_n, \; \xi_1 \cdots \xi_m); \; w' = w'(i_1 \cdots i_n, \; \xi_1 \cdots \xi_m)$$

The magnetic force is the negative partial derivative of the magnetic energy with respect to the appropriate associated displacement, with the other displacements and all of the flux linkages held constant. Similarly, the force can be found from the coenergy function by taking the derivative with respect to the associated displacement with the other displacements and the currents held constant.

3.6 Polarization and Magnetization Force Densities on Tenuous Dipoles

Forces due to polarization and magnetization lend further emphasis to the importance of making a distinction between forces on microscopic charged particles and macroscopic forces on materials supporting those charges. The experiment depicted by Fig. 3.6.1 makes it clear that (1) there is more to the force density than accounted for by the Lorentz force density, and (2) the additional force density is not $\rho_p\vec{E}$ (or in the magnetic analogue, $\rho_m\vec{H}$).

A pair of capacitor plates are dipped into a dielectric liquid. With the application of a potential difference v, it is found experimentally that the liquid rises between the plates.* To make it clear that the issues involved can be understood in terms of lumped-parameter concepts, the liquid between the plates is replaced by a solid dielectric material having the same polarizability as the liquid, so that the problem is reduced to one of a solid dielectric slab rising between the plates as it is pulled from the liquid below.

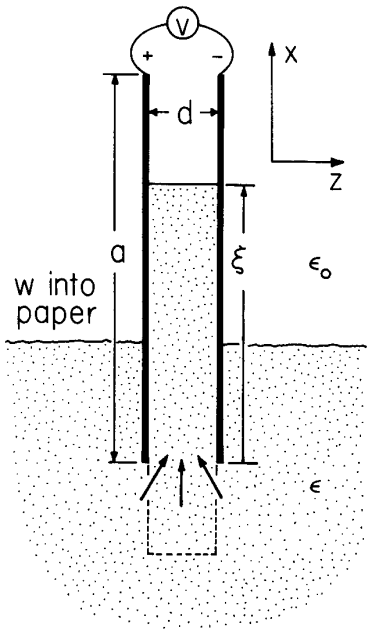

Fig. 3.6.1. Experiment demonstrating the existence of polarization forces that are not explicable in terms of forces on single charges.

Recall that if the interface is well removed from the edges of the plates, an exact solution satisfying the quasistatic differential equations and boundary conditions in the neighborhood of the interface is $\vec{E} = (v/d)\vec{i}_z$. Of course, there is a fringing field in the neighborhood of the edges of the capacitor plates. However, because the slab and the liquid have the same dielectric constant and $\rho_f = 0$, the fringing field has the same distribution as if the dielectric were not present.

It might be tempting to take the force as being the product of the net charge at any given point and the local electric field, or $\rho_p\vec{E}$. However, everywhere in the dielectric bulk the polarization density is proportional by the same constant to the electric field (Eq. 2.16.1). Because $\rho_f = 0$, it follows from Gauss' law that $\vec{E}$ and hence $\vec{P}$ have no divergence, and so there is also no polarization charge in the dielectric. Furthermore, because the electric field is uniform and tangential to the interface, there is not even a polarization surface charge density at the interface (Eq. 2.10.21). Throughout the dielectric, on the interface and in the bulk, there is no polarization charge. Clearly, the force which makes the dielectric rise between the plates cannot be accounted for by a polarization charge density.

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* In an experiment, a-c voltage is used with a sufficiently high frequency that the material responds only to the rms field and free charge cannot accumulate in the bulk.

If the polarized material is composed of individual dipoles, each subject to an electrical force, and each transmitting this electrical force to the neutral medium, it is clear that there is really no reason to expect that the force density should take the same form as that for free charges. With free charges, it is the individual charges that transmit their forces to the neutral medium through mechanisms discussed in Sec. 3.2. Now concern is with the force on individual dipoles which transmit that force to the neutral medium, either because they are tied to a lattice structure (Fig. 2.8.1) or through collisional mechanisms similar to those discussed for charge carriers in Sec. 3.2.

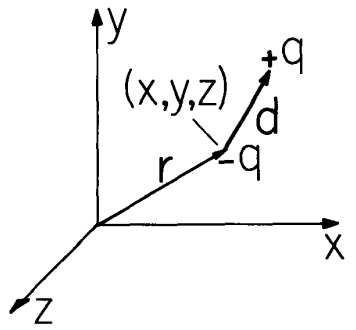

Fig. 3.6.2. Definition of displacement and charge locations for dipole.

In the following, it is assumed that the dipoles are subject to an electric field that is the average, or macroscopic, electric field. The development ignores the distortion of the electric field intensity at one dipole because of the neighboring dipoles. For this reason, the result is designated a force density acting on tenuous dipoles.

A single dipole is shown in Fig. 3.6.2. The dipole can be pictured as a pair of oppositely signed charges having the vector separation $\vec{d}$. The negative charge is located at $\vec{r}$. With the assumption that the force on the dipole is transmitted to the medium, the procedure is to compute the force on a single dipole, and then to average this force over all the dipoles. The net force in the ith direction on the pair of charges taken as a unit is

$$f_i = \lim_{\substack{\vec{d}\to 0 \\ q\to\infty}} q[E_i(\vec{r}+\vec{d}) - E_i(\vec{r})] \tag{2}$$

The limit is one in which the spacing of the charges becomes extremely small compared to other distances of interest and, at the same time, the magnitude of the charges becomes very large, so that the product $q\vec{d} \equiv \vec{p}$ remains finite. The dipole moment is defined as $\vec{p}$. The required limit of Eq. 2 becomes

$$f_i = \lim_{\substack{\vec{d}\to 0 \\ q\to\infty}} q[E_i(\vec{r}) + \frac{\partial E_i}{\partial x_j} d_j - E_i(\vec{r})] = p_j \frac{\partial E_i}{\partial x_j} \tag{3}$$

Thus, there is a net force on each dipole given in vector notation by

$$\vec{f} = \vec{p}\cdot\nabla\vec{E} \tag{4}$$

Note that implicit to this vector representation is the definition of what is meant by the operator $\vec{A}\cdot\nabla\vec{B}$

By assumption, the net force on each dipole is transmitted to the macroscopic medium and it is appropriate then to think of averaging these polarization forces over all dipoles within the medium. In general, this average would have to be taken with recognition that the microscopic dipoles could assume a spectrum of polarizations in a given electric field intensity. For present purposes, the average can simply be represented as the multiplication of Eq. 4 by the number of dipoles, n, per unit volume. With the definition of the polarization density as $\vec{P} = n\vec{p}$, the Kelvin polarization force density is found:

$$\vec{F} = \vec{P}\cdot\nabla\vec{E} \tag{5}$$

Can the force density given by Eq. 5 be used to explain the rise of the dielectric between the plates in Fig. 3.6.1? Certainly, there is no force density in material regions of uniform electric field, because then the spatial derivatives called for with Eq. 5 vanish. However, in the fringing field at the lower edges of the plates, the electric field intensity does vary rapidly. In that region, the permittivity is a constant, and for a linear dielectric, where $\vec{D} = \varepsilon\vec{E}$, Eq. 5 becomes [in dealing with vectors and tensors, a term in which a subscript appears twice is to be summed 1 to 3 (unless otherwise indicated)]

$$F_i = (\varepsilon - \varepsilon_o)E_j \frac{\partial E_i}{\partial x_j} = (\varepsilon - \varepsilon_o)E_j \frac{\partial E_j}{\partial x_i} = (\varepsilon - \varepsilon_o)\frac{\partial}{\partial x_i}(\frac{1}{2} E_j E_j) \tag{6}$$

where the irrotational nature of $\vec{E}$ is exploited, $\partial E_i/\partial x_j = \partial E_j/\partial x_i$. In vector notation, Eq. 6 becomes

$$\vec{F} = \nabla[\frac{1}{2}(\varepsilon - \varepsilon_o)\vec{E}\cdot\vec{E}] \tag{7}$$

Remember, this relation pertains only to regions of a linear dielectric in which the permittivity is constant, and is simply a means of visualizing the distribution of the Kelvin force density. In such regions, the force density has the direction of maximum rate of increase of the electric energy storage. Typical force vectors, sketched in Fig. 3.6.1, tend to push the dielectric upward between the plates. It is important not to overgeneralize from Eq. 7. In any configuration in which there is a component of $\vec{E}$ perpendicular to an interface, there is a singular component of the Kelvin force density acting at the interface -- a surface force density. Such a component would be incorrectly inferred from Eq. 7, which is not valid through the interfacial region.

Consider now the force density acting on a continuum of dilute magnetic dipoles that, like the analogous electric dipoles just considered, pass along a force of electric origin to a macroscopic medium via collisions or lattice constraints. It is not possible to use the Lorentz force law as a starting point unless magnetic monopoles and an analogous force law on these magnetic "charges" is postulated. Without introducing such notions, the Kelvin magnetization force density can be deduced as follows.

Electroquasistatic and magnetoquasistatic systems are pictured abstractly in Fig. 3.6.3. A volume enclosing the region occupied by a dipole having the position $\vec{\xi}$ has a surface S and includes neither free charge in the EQS system nor free current in the MQS system. Hence the fields are governed by

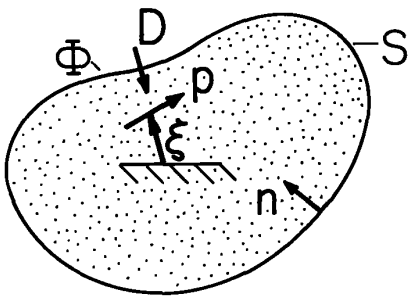

Fig. 3.6.3a. EQS system

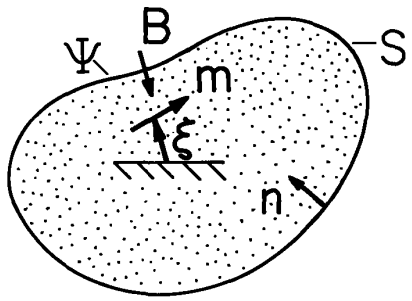

Fig. 3.6.3b. MQS system

$$\nabla \times \vec{E} = 0;\ \vec{E} = -\nabla\Phi \qquad \nabla \times \vec{H} = 0;\ \vec{H} = -\nabla\Psi \qquad (8)$$

$$\nabla\cdot(\varepsilon_o\vec{E} + \vec{P}) = 0;\ \vec{P} = n\vec{p} \qquad \nabla\cdot(\mu_o\vec{H} + \mu_o\vec{M}) = 0;\ \vec{M} = n\vec{m} \qquad (9)$$

Statements that the input of electric energy either goes into increasing the total energy stored or into doing work on the dipoles are (see Eqs. 3.5.1 and 2.13.4 or Eq. 3.5.10 and Eq. 2.14.9 integrated by parts):

$$\oint_S \Phi\delta\vec{D}\cdot\vec{n}da = \delta w + \vec{f}\cdot\delta\vec{\xi} \qquad \oint_S \Psi\delta\vec{B}\cdot\vec{n}da = \delta w + \vec{f}\cdot\delta\vec{\xi} \qquad (10)$$

To find the force on the dipole, the energy would be determined as a function of the electrical excitations and $\vec{\xi}$. Then, with the understanding that the derivative is taken with the quantities $\vec{D}\cdot\vec{n}$ and $\vec{B}\cdot\vec{n}$, respectively, held fixed on the surface S, the respective forces follow as

$$f_i = -\frac{\partial w}{\partial \xi_i} \qquad f_i = -\frac{\partial w}{\partial \xi_i} \qquad (11)$$

Now, what would be obtained if this procedure were carried through for the electric case is already known to be given by Eq. 4. Moreover, there is a complete analogy between every aspect of the electric and magnetic systems. The calculation in the magnetic case need not be repeated once the electric one is carried out. Rather, an identification of variables suffices to give the answer, $\vec{E} \to \vec{H}$, $\vec{P} \to \mu_o\vec{M}$. Hence, it follows that Eq. 5 is replaced by the Kelvin magnetization force density

$$\vec{F} = \mu_o\vec{M}\cdot\nabla\vec{H} \qquad (12)$$

The Kelvin force densities, Eqs. 5 and 12, suffer the weakness that they do not take into account the interaction between dipoles. Moreover, is the average over the spectrum of dipole moments $\vec{p}$ or $\vec{m}$ leading to the polarization and magnetization densities consistent with the usage of these densities in Chap. 2? These difficulties are overcome by a derivation based on thermodynamic principles. Because force densities are then based on electrically measured constitutive laws, consistency with definitions already introduced is insured.

3.7 Electric Korteweg-Helmholz Force Density

The thermodynamic technique used in this section for deducing the electric force density with combined effects of free charge and polarizarion is a generalization of that used in determining discrete forces in Sec. 3.5. This principle of virtual work is exploited because it is not practical to predict the relationship between microscopic and macroscopic fields.

In any derivation of a force density, it is important to be clear about (a) what empirically determined information is required, and (b) what postulates or assumptions are incorporated into the derivation or are implicit to an application of the force density. Generally, empirically determined information can be used to replace assumptions. As derived here, the only empirical information required is an electrical constitutive law relating the macroscopic electric field to the polarization density $\vec{P}$ (or displacement $\vec{D}$). This relationship is typically determined by making electrical measurements on homogeneous samples of the material. These amount to measurements of the terminal characteristics of capacitor-like configurations incorporating samples of the material. (In the lumped-parameter systems of Sec. 3.5, the analogous empirical information was the electrical terminal relation.) With so little empirical information, the force density can only be identified if the system considered is a conservative thermodynamic subsystem. Thus, the force density is derived picturing the system as having no dissipation mechanisms. (The same conservative system is considered in Sec. 3.5 to find discrete forces.) The assumption is then made that the force density remains valid even in modeling systems with dissipation. If dissipation mechanisms were to be incorporated into the system considered, then a virtual power principle could be exploited to find the force density, but additional empirical information would be required.

Experiments show that, for a wide range of materials, electrical constitutive laws take the form of state functions

$$\vec{E} = \vec{E}(\alpha_1 \cdots \alpha_m, \vec{D}) \quad \text{or} \quad \vec{D} = \vec{D}(\alpha_1 \cdots \alpha_m, \vec{E}) \tag{1}$$

The α's are properties of the material. Thus, if measurements are made on a homogeneous sample of the material, the α's are varied by changing the composition of the sample. For example, α_1 might be the concentration of dipoles of a given species, or the concentration of one liquid in another. The number of α's used depends on the specific application. Most important for now is the distinction between changing $\vec{E}$ in Eq. 1 by changing the material and hence changing α's, and doing so by changing $\vec{D}$. Some special cases of Eq. 1 are given in Table 3.7.1.

Table 3.7.1. Constitutive laws having the general form of Eq. 1a.

Law	Description
$\vec{E} = \varepsilon^{-1}(\alpha_1 \cdots \alpha_m)\vec{D}$	Electrically linear and (fields) collinear
$E_i = s_{ij}(\alpha_1 \cdots \alpha_m)D_j$	Electrically linear and anisotropic
$\vec{E} = \varepsilon^{-1}(\alpha_1 \cdots \alpha_m, D^2)\vec{D}$	Electrically nonlinear and (fields) collinear
$E_i = s_{ij}(\alpha_1 \cdots \alpha_m, D_1, D_2, D_3)D_j$	Electrically nonlinear and anisotropic

The third case of the table might represent a material in which dipoles are in Brownian equilibrium with a nonpolar liquid. An applied field tends to line up the dipoles and hence give rise to a polarization density and hence to a contribution to $\vec{D}$. In terms of two properties (α_1, α_2), a model including the saturation effect, resulting as all dipoles become aligned with the field, might be

$$\varepsilon = \frac{\alpha_1}{\sqrt{1 + \alpha_2^2 \vec{E}\cdot\vec{E}}} + \varepsilon_o \tag{2}$$

Built into this example, and the general relation, Eq. 1, is the assumption that the constitutive law is a state function. It does not depend on rates of change, and it is a single-valued function of the variables and hence not dependent on the path followed to arrive at the given state.

The continuum now considered is not homogeneous, in that at any given instant the α's can vary from one position to another. Moreover, for the electromechanical subsystem considered, the properties are tied to the material. As the material moves, properties change. For material within a volume of fixed identity,

$$\int_V \alpha_i dV = \text{constant} \tag{3}$$

By definition, the volume V is always composed of the same material. By definition, the α's must satisfy Eq. 3 when the subsystem is considered to be isolated from other subsystems.

The finite number of mechanical degrees of freedom for the discrete coupling of Sec. 3.5 is now replaced by an infinite number of degrees of freedom. The mechanical continuum, perhaps a fluid, perhaps a solid, is capable of undergoing the vector deformations $\delta\vec{\xi}$. These incremental displacements are viewed as small departures from an equilibrium mechanical configuration which is precisely that for which the force density is required.

Since the time derivative of Eq. 3 vanishes, the generalized Leibnitz rule, Eq. 2.6.5, gives

$$\frac{d}{dt}\int_V \alpha_i dV = \int_V \frac{\partial \alpha_i}{\partial t} dV + \oint_S \alpha_i \frac{\partial \vec{\xi}}{\partial t}\cdot\vec{n}da = 0 \tag{4}$$

where by definition the velocity of the surface S is equal to that of the material ($\vec{v}_s \rightarrow \frac{\partial \vec{\xi}}{\partial t}$). Gauss' theorem converts the second integral to a volume integral. Although of fixed identity, the volume is arbitrary, and so it follows from Eq. 4 that changes in the property α_i are linked to the material deformations by an expression that is equivalent to Eq. 3:

$$\delta\alpha_i = -\nabla\cdot(\alpha_i\delta\vec{\xi}) \tag{5}$$

The framework has now been established for stating and exploiting conservation of energy for the electromechanical subsystem. The procedure is familiar from Sec. 3.5. With electrical excitations absent, a system, such as shown in Fig. 2.13.1, is assembled mechanically. Because the force density of electrical origin is by definition zero during the process, no work is required. The system now consists of rigid electrodes for producing part or all of the electrical excitations and a mechanical continuum in the intervening space. This material is described by Eq. 1. With the mechanical deformations fixed ($\delta\vec{\xi} = 0$), the electrical excitations are next raised by placing bulk charges at the positions of interest in the material and by raising the potentials on the electrodes. The result is a stored electrical energy given by Eq. 2.13.6:

$$w = \int_V W dV;\quad W = \int_0^{\vec{D}} \vec{E}(\alpha_1\cdots\alpha_m,\vec{D}')\cdot\delta\vec{D}' \tag{6}$$

Here, V is the volume occupied by the material and the fields, and hence excluding the electrodes.

Now, with the net charge on each electrode constrained to be constant, consider variations in the energy caused by incremental displacements of the material. A statement of energy conservation accounting for work done on the external mechanical world by the force density of electrical origin is

$$\int_V [\delta W + \vec{F}\cdot\delta\vec{\xi}]dV = 0 \tag{7}$$

There are two consequences of the incremental displacement. First, the mechanical deformation carries the properties with it, as already stated by Eq. 5. Second, there is a redistribution of the free charge. Because the system is conservative, the free charge is constrained to move with the material. The charge within a volume always composed of the same material particles is constant. Thus, Eq. 3 also holds with $\alpha_i \rightarrow \rho_f$, and it follows that an expression similar to Eq. 5 can be written for the change in charge density at a given location caused by the material displacement $\delta\vec{\xi}$:

$$\delta\rho_f = -\nabla\cdot(\rho_f\delta\vec{\xi}) \tag{8}$$

It is extremely important to recognize the difference between δW in Eq. 7, and δW in Sec. 2.13. In Eq. 7, the change in energy is caused by material displacements $\delta\vec{\xi}$, whereas in Sec. 2.13 it is due to changes in the electrical excitations. The energy W is assumed to be a state function of the same variables as used to express the constitutive law, Eq. 1. Hence,

$$\delta W = \sum_{i=1}^{m} \frac{\partial W}{\partial \alpha_i}\delta\alpha_i + \frac{\partial W}{\partial \vec{D}}\cdot\delta\vec{D} \tag{9}$$

where

$$\frac{\partial W}{\partial \vec{D}}\cdot\delta\vec{D} \equiv \sum_{i=1}^{3} \frac{\partial W}{\partial D_i}\delta D_i$$

With the understanding that the partial derivative is taken with the α's held fixed, it follows from Eq. 6 that

$$\frac{\partial W}{\partial D_j} = E_j \tag{10}$$

Hence, the last term in Eq. 9 is written using Eq. 10 with $\vec{E}$ in turn replaced by $-\nabla\Phi$. Then, integration by parts* gives

$$\int_V \frac{\partial W}{\partial \vec{D}} \cdot \delta\vec{D}dV = -\oint_S \Phi\delta\vec{D}\cdot\vec{n}da + \int_V \Phi(\nabla\cdot\delta\vec{D})dV \tag{11}$$

The part of the surface coincident with the electrode surfaces gives a contribution from each electrode equal to the electrode potential multiplied by the change in electrode charge. Because the electrode charges are held fixed while the material is deformed, this integration gives no contribution. The remaining part of the surface integration is sufficiently well removed from the region of interest that the fields have fallen off sufficiently to make a negligible contribution. Thus, the first term on the right vanishes and, because of Gauss' law, Eq. 11 becomes

$$\int \frac{\partial W}{\partial \vec{D}} \cdot \delta\vec{D}dV = \int \Phi\delta\rho_f dV \tag{12}$$

It is now possible to write Eq. 7 with effects of $\delta\vec{\xi}$ represented explicitly. Substitution of Eq. 8 into 12 and then Eqs. 12 and 5 into 9, and finally of Eq. 9 into 7, gives

$$\int_V [-\sum_{i=1}^{m} \frac{\partial W}{\partial \alpha_i} \nabla\cdot(\alpha_i\delta\vec{\xi}) - \Phi\nabla\cdot(\rho_f\delta\vec{\xi}) + \vec{F}\cdot\delta\vec{\xi}]dV = 0 \tag{13}$$

With the objective of writing the integrand in the form $(\ )\cdot\delta\vec{\xi}$, the first two terms are integrated by parts. Because the surface integrations are either on the rigid electrode surfaces where $\delta\vec{\xi}\cdot\vec{n} = 0$, or at infinity where the fields have decayed to zero, and $\vec{E} = -\nabla\Phi$, Eq. 13 becomes

$$\int_V [\sum_{i=1}^{m} \alpha_i \nabla(\frac{\partial W}{\partial \alpha_i}) - \rho_f\vec{E} + \vec{F}]\cdot\delta\vec{\xi}dV = 0 \tag{14}$$

It is tempting, and in fact correct, to set the integrand of this expression to zero. But the justification is not that the volume V is arbitrary. To the contrary, the volume V is a special one enclosing all of the region occupied by the deformable medium and fields. (The volume integration plays the role of a summation over the mechanical variables for the lumped-parameter systems of Sec. 3.5.) The integrand is zero because $\delta\vec{\xi}$ (like the lumped-parameter displacements) is an independent variable. The equation must hold for any deformation, including one confined to any region where $\vec{F}$ is to be evaluated:

$$\vec{F} = \rho_f\vec{E} - \sum_{i=1}^{m} \alpha_i \nabla(\frac{\partial W}{\partial \alpha_i}) \tag{15}$$

It is most often convenient to write the second term so that it is clear that it consists of a force density concentrated where there are property gradients and the "gradient of a pressure":

$$\vec{F} = \rho_f\vec{E} + \sum_{i=1}^{m} \frac{\partial W}{\partial \alpha_i} \nabla\alpha_i - \nabla[\sum_{i=1}^{m} \alpha_i \frac{\partial W}{\partial \alpha_i}] \tag{16}$$

The implications of Eq. 16 and the method of its derivation are appreciated by considering three commonly encountered limiting cases and then writing Eq. 16 in such a way that its relation to the Kelvin force density is clear.

Incompressible Media: Deformations are then such that

$$\nabla\cdot\delta\vec{\xi} = 0 \tag{17}$$

Because $\delta\vec{\xi}\cdot\vec{n} = 0$ on the rigid electrode surfaces that comprise part of the surface S enclosing V in Eq. 7, any pressure function π that approaches zero with sufficient rapidity at infinity to make the surface integration there negligible will satisfy the relation

---

*Integration by parts in three dimensions amounts to

$$\int_V \Psi\nabla\cdot\vec{A}dV = \int_V \nabla\cdot(\Psi\vec{A})dV - \int_V \vec{A}\cdot\nabla\Psi dV = \oint_S \Psi\vec{A}\cdot\vec{n}da - \int_V \vec{A}\cdot\nabla\Psi dV$$

$$\oint_S \pi\delta\vec{\xi}\cdot\vec{n}da = \int_V \nabla\cdot(\pi\delta\vec{\xi})dV = 0 \tag{18}$$

Thus, Eq. 14 remains valid even if the volume integral of Eq. 18 is added to it. But, for incompressible deformations as defined with Eq. 17, $\nabla\cdot(\pi\delta\vec{\xi}) = \nabla\pi\cdot\delta\vec{\xi}$. Thus, the term added to Eq. 14, like those already appearing in its integrand, can be written with $\delta\vec{\xi}$ as a factor. It follows that for incompressible deformations, the gradient of any scalar pressure, π, can be added to the force density of Eq. 16. For example, π might be $\vec{P}\cdot\vec{E}$, since this function decays with distance from the system sufficiently rapidly to make the contribution of the surface integration at infinity vanish. On the basis of this apparent arbitrariness in the force density, the following observation is now made for the first time, and will be emphasized again in Chap. 8. Two force densities differing by the gradient of a scalar pressure will give rise to the same incompressible deformations. Physically this is so because in modeling a continuum as incompressible, the pressure becomes a "left-over" variable. It becomes whatever it must be to make Eq. 17 valid. Whatever the $\nabla\pi$ added to the force density of electrical origin, π can be absorbed into the "mechanical" pressure of the continuum-force equation.

For incompressible deformations, where the force density is arbitrary to within the gradient of a pressure, the gradient term can be omitted from Eq. 16, which then takes the convenient form

$$\vec{F} = \rho_f\vec{E} + \sum_{i=1}^{m} \frac{\partial W}{\partial \alpha_i}\nabla\alpha_i \tag{19}$$

This expression concentrates the force density where there are property gradients. In a charge-free system composed of regions having uniform properties, the force density is thus confined to interfaces between regions.

Incompressible and Electrically Linear: For an incompressible material having the constitutive law

$$\vec{D} = \varepsilon_o(1 + \chi_e)\vec{E} \equiv \varepsilon\vec{E} \tag{20}$$

the susceptibility χ_e is conserved by a volume of fixed identity. That is, α_1 can be taken as χ_e in Eq. 3 and m = 1. Then, from Eq. 6,

$$W = \frac{1}{2}\frac{D^2}{\varepsilon_o(1 + \chi_e)}; \quad \frac{\partial W}{\partial \chi_e} = -\frac{\varepsilon_o}{2}E^2 \tag{21}$$

and because $\nabla\chi_e = \nabla[(1 + \chi_e)]$, it follows that the force density of Eq. 19 specializes to

$$\vec{F} = \rho_f\vec{E} - \frac{1}{2}E^2\nabla\varepsilon \tag{22}$$

Electrically Linear with Polarization Dependent on Mass Density Alone: Certainly a possible parameter α_1 is the mass density ρ, since then Eq. 3 is satisfied. For a compressible medium it is possible that the susceptibility χ_e in Eq. 20 is only a function of ρ. Then,

$$\alpha_1 = \rho, \ \chi_e = \chi_e(\rho), \ W = \frac{1}{2}\frac{D^2}{\varepsilon_o[1 + \chi_e(\rho)]}; \quad \frac{\partial W}{\partial \rho} = -\frac{\varepsilon_o}{2}E^2\frac{\partial \chi_e}{\partial \rho} \tag{23}$$

and, because $(\partial\varepsilon/\partial\rho)\nabla\rho = \nabla\varepsilon$, the force density given by Eq. 16 becomes

$$\vec{F} = \rho_f\vec{E} - \frac{1}{2}E^2\nabla\varepsilon + \nabla[\frac{1}{2}\rho\frac{\partial\varepsilon}{\partial\rho}E^2] \tag{24}$$

Because the last term is associated with volumetric changes in the material, it is called the electrostriction force density.

Relation to the Kelvin Force Density: Because $W = W(\alpha_1,\alpha_2\cdots\alpha_m,\vec{D})$, the kth component of the gradient of W is

$$(\nabla W)_k = \sum_{i=1}^{m}\frac{\partial W}{\partial \alpha_i}\frac{\partial \alpha_i}{\partial x_k} + \frac{\partial W}{\partial D_j}\frac{\partial D_j}{\partial x_k} \tag{25}$$

In view of Eq. 10, it follows that

$$\sum_{i=1}^{m}\frac{\partial W}{\partial \alpha_i}\frac{\partial \alpha_i}{\partial x_k} = \frac{\partial W}{\partial x_k} - \frac{\partial}{\partial x_k}(\vec{E}\cdot\vec{D}) + D_j\frac{\partial E_j}{\partial x_k} \tag{26}$$

This expression can be substituted for the second term in Eq. 16, which with some manipulation then becomes

$$\vec{F} = \rho_f \vec{E} + \vec{P}\cdot\nabla\vec{E} + \nabla[\frac{1}{2}\varepsilon_o \vec{E}\cdot\vec{E} + W - \vec{E}\cdot\vec{D} - \sum_{i=1}^{m} \alpha_i \frac{\partial W}{\partial \alpha_i}] \tag{27}$$

In this form, the force density is the sum of a free charge force density, the Kelvin force density (Eq. 3.6.5) and the gradient of a pressure. This last term can consistently be ignored in predicting the deformations of an incompressible continuum. For such situations, the Kelvin force density or the Korteweg-Helmholtz force density in the form of Eq. 19 will give rise to the same deformations. Note that they have very different distributions.

Apparently the last term in Eq. 27 represents the interaction between dipoles omitted from the derivation of the Kelvin force density. In fact, this term vanishes when the constitutive law takes a form consistent with the polarization being due to noninteracting dipoles. In that case, the susceptibility should be linear in the mass density so that $\chi_e = c\rho$, where c is a constant. In Eq. 23, $\partial\chi_e/\partial\rho = c$, and evaluation shows that, indeed, the last term in Eq. 27 does vanish.

3.8 Magnetic Korteweg-Helmholtz Force Density

Thermodynamic techniques for determining the magnetization force density are analogous to those outlined for the polarization force density in Sec. 3.7. In fact, if there were no free current density, the magnetic field intensity, like the electric field intensity, would be irrotational. It would then be possible to make a derivation that would be the complete analog of that for the polarization force density. However, in the following the force density due to free currents is included and hence $\vec{H}$ is not irrotational.

The constitutive law takes the form

$$\vec{H} = \vec{H}(\alpha_1,\alpha_2\cdots\alpha_m,\vec{B}) \text{ or } \vec{B} = \vec{B}(\alpha_1,\alpha_2\cdots\alpha_m,\vec{H}) \tag{1}$$

with specific possibilities given in Table 3.7.1 with $\varepsilon \to \mu$, $\vec{E} \to \vec{H}$ and $\vec{D} \to \vec{B}$. A conservative electromechanical subsystem is assembled mechanically, with no electrical excitations, so that it assumes a configuration identical to the one for which the force density is required. By the definition of the subsystem, this process requires no energy. Then, with the mechanical system fixed (the α's fixed), electrical excitations are applied so as to establish the free currents in excitation coils and in the medium itself, with the distribution that for which the force density is required. This procedure is formalized in Sec. 2.12 and a system schematic is shown in Fig. 2.14.2. As was shown in Sec. 2.14, currents in excitation coils are conveniently regarded as part of the total distribution of free current density. Hence, the volume of interest now includes all of the region permeated by the magnetic field.

Now, with the electrical excitations established, a statement of conservation of energy, with the electrical excitations held fixed but the material undergoing an incremental displacement, is Eq. 3.7.7, where now W is the magnetic energy density given from Eq. 2.14.10 by

$$W = \int_0^{\vec{B}} \vec{H}(\alpha_1,\alpha_2\cdots\alpha_m,\vec{B}')\cdot\delta\vec{B}' \tag{2}$$

The following steps, leading to a deduction of the force density, are analogous to those taken in Sec. 3.7. The link between the α's and $\delta\vec{\xi}$ is given by Eq. 3.7.5. What is the connection between $\vec{J}_f$ and $\delta\vec{\xi}$?

Actually, it is a link between the flux linkage and $\vec{\xi}$ that is appropriate. If the medium is to both support a free current density and be conservative, the material must be idealized as having an infinite conductivity. This means that any open material surface S (surface of fixed identity) must link a constant flux:

$$\delta\int_S \vec{B}\cdot\vec{n}da = 0 \tag{3}$$

One way to make this deduction is to use the integral form of Faraday's law for a contour C enclosing a surface S of fixed identity, Eq. 2.7.3b, with $\vec{v} = \vec{v}_s$. Because the medium is perfectly conducting, $\vec{E}' = 0$ and what remains of Faraday's law is Eq. 3. From the generalized Leibnitz rule, Eq. 2.6.4, Eq. 3 and the solenoidal nature of $\vec{B}$ require that

$$\int_S \delta\vec{B}\cdot\vec{n}da + \oint_C (\vec{B}\times\delta\vec{\xi})\cdot d\vec{\ell} = 0 \tag{4}$$

Stokes's theorem, Eq. 2.6.3, converts the contour integral to a surface integral. Because this surface is arbitrary, the sum of the integrands must vanish. If it is further recognized that $\delta\vec{B} = \nabla \times \delta\vec{A}$, then it follows that

$$\delta\vec{A} = \delta\vec{\xi} \times \vec{B} \tag{5}$$

Thus, there is established the link between material deformations and the alterations of the field that are required if the deformations are to be flux-conserving.

The change in W associated with the material deformation, called for in the conservation of energy equation, Eq. 3.7.7, is in general

$$\delta W = \sum_{i=1}^{n} \frac{\partial W}{\partial \alpha_i} \delta\alpha_i + \frac{\partial W}{\partial \vec{B}} \cdot \delta\vec{B} \tag{6}$$

where, in view of Eq. 2,

$$\frac{\partial W}{\partial B_j} = H_j \tag{7}$$

It is the integral over the total volume V of δW that is of interest. The integral of the last term in Eq. 6 is

$$\int_V \frac{\partial W}{\partial \vec{B}} \cdot \delta\vec{B} dV = \int_V \vec{H} \cdot \delta\vec{B} dV = \int_V \vec{H} \cdot \nabla \times \delta\vec{A} dV \tag{8}$$

Because the fields decay to zero sufficiently rapidly at infinity that the surface integral vanishes and because Ampere's law, Eq. 2.3.23b, gives $\nabla \times \vec{H} = \vec{J}_f$, integration of the last term in Eq. 8 by parts gives

$$\int_V \frac{\partial W}{\partial \vec{B}} \cdot \delta\vec{B} dV = \int_V \nabla \cdot (\delta\vec{A} \times \vec{H}) dV + \int_V \delta\vec{A} \cdot \nabla \times \vec{H} dV = \oint_S \delta\vec{A} \times \vec{H} \cdot \vec{n} da + \int_V \vec{J} \cdot \delta\vec{A} dV = \int_V \delta\vec{A} \cdot \vec{J}_f dV \tag{9}$$

Substitution for $\delta\vec{A}$ from Eq. 5 finally gives an expression explicitly showing the $\vec{\xi}$ dependence:

$$\int_V \frac{\partial W}{\partial \vec{B}} \cdot \delta\vec{B} dV = \int_V \delta\vec{\xi} \times \vec{B} \cdot \vec{J}_f dV = -\int_V \vec{J}_f \times \vec{B} \cdot \delta\vec{\xi} dV \tag{10}$$

Finally, the energy conservation statement, Eq. 3.7.7, is written with δW given by Eq. 6 and in turn, $\delta\alpha_i$ given by Eq. 3.7.5 and the last term given by Eq. 10:

$$\int_V [-\sum_{i=1}^{n} \frac{\partial W}{\partial \alpha_i} \nabla \cdot (\delta\vec{\xi}\alpha_i) - \vec{J}_f \times \vec{B} \cdot \delta\vec{\xi} + \vec{F} \cdot \delta\vec{\xi}] dV = 0 \tag{11}$$

With the objective of writing the first term as a dot product with $\delta\vec{\xi}$, the first term is integrated by parts (exactly as in going from Eq. 3.7.13 to Eq. 3.7.14) to obtain

$$\int_V [\sum_{i=1}^{n} \alpha_i \nabla \frac{\partial W}{\partial \alpha_i} - \vec{J}_f \times \vec{B} + \vec{F}] \cdot \delta\vec{\xi} dV = 0 \tag{12}$$

The integrand must be zero, not because the volume is arbitrary (it includes all of the system involved in the electromechanics) but rather because the virtual displacements $\delta\vec{\xi}$ are arbitrary in their distribution. Hence, the force density is

$$\vec{F} = \vec{J}_f \times \vec{B} - \sum_{i=1}^{n} \alpha_i \nabla \frac{\partial W}{\partial \alpha_i} \tag{13}$$

The special cases considered in Sec. 3.7 have analogs that similarly follow from Eq. 13. Because what is involved in deriving these forms involves the magnetization term in Eq. 13, and not the free current force density, these expressions can be written down by direct analogy.

Incompressible Media: The convenient form emphasizing the importance of regions where there are property gradients is

$$\vec{F} = \vec{J}_f \times \vec{B} + \sum_{i=1}^{n} \frac{\partial W}{\partial \alpha_i} \nabla \alpha_i \tag{14}$$

Incompressible and Electrically Linear: With a constitutive law

$$\vec{B} = \mu_o(1 + \chi_m)\vec{H} = \mu\vec{H} \tag{15}$$

the force density of Eq. 13 reduces to

$$\vec{F} = \vec{J}_f \times \vec{B} - \frac{1}{2} H^2 \nabla\mu \tag{16}$$

Electrically Linear with Magnetization Dependent on Mass Density Alone: With the constitutive law in the form of Eq. 15, but $\chi_m = \chi_m(\rho)$, where ρ is the mass density, the force density is the sum of Eq. 14 and a magnetostrictive force density taking the form of the gradient of a pressure:

$$\vec{F} = \vec{J}_f \times \vec{B} - \frac{1}{2} H^2 \nabla\mu + \nabla\left(\frac{1}{2} \rho \frac{\partial \mu}{\partial \rho} H^2\right) \tag{17}$$

Relation to Kelvin Force Density: With the stipulation that $W = W(\alpha_1, \alpha_2 \cdots \alpha_m, \vec{B})$ is a state function, Eq. 13 becomes the sum of a Lorentz force density due to the free current density, the Kelvin force density and the gradient of a pressure:

$$\vec{F} = \vec{J}_f \times \mu_o \vec{H} + \mu_o \vec{M} \cdot \nabla \vec{H} + \nabla\left[\frac{1}{2} \mu_o \vec{H} \cdot \vec{H} + W - \vec{H} \cdot \vec{B} - \sum_{i=1}^{m} \alpha_i \frac{\partial W}{\partial \alpha_i}\right] \tag{18}$$

The discussion of Sec. 3.7 is as appropriate for understanding these various forms of the magnetic force density as it is for the electric force density.

3.9 Stress Tensors

Most of the force densities of concern in this text can be written as the divergence of a stress tensor. The representation of forces in terms of stresses will be used over and over again in the chapters which follow. This section is intended to give a brief summary of the differential and integral properties of the stress tensor.

Suppose that the ith component of a force density can be written in the form

$$F_i = \frac{\partial T_{ij}}{\partial x_j}; \quad (\vec{F} = \nabla \cdot \vec{\vec{T}}) \tag{1}$$

Here, the Einstein summation convection is applicable, so that because the j's appear twice in the same term, they are to be summed from one to three. An alternative notation, in parentheses, represents the same operation in vector notation. Much of the convenience of recognizing the stress tensor representation of a force density comes from then being able to convert an integration of the force density over a volume to an integration of the stress tensor over a surface enclosing the volume. This generalization of Gauss' theorem is easily shown by fixing attention on the ith component (think of i as given) and defining a vector such that

$$\vec{G}_i = T_{i1}\vec{i}_1 + T_{i2}\vec{i}_2 + T_{i3}\vec{i}_3 \tag{2}$$

Then the right-hand side of Eq. 1 is simply the divergence of $\vec{G}_i$. Gauss' theorem then shows that

$$\int_V F_i dV = \int_V \nabla \cdot \vec{G}_i dV = \oint_S \vec{G}_i \cdot \vec{n} da \tag{3}$$

or, in index notation and using the definition of $\vec{G}_i$ from Eq. 2,

$$\int_V F_i dV = \oint_S T_{ij} n_j da \tag{4}$$

This tensor form of Gauss' theorem is the integral counterpart of Eq. 1. Physically, Eq. 4 states that an alternative to integrating the force density in some Cartesian direction over the volume V is an integration of the integrand on the right over a surface completely enclosing that volume V. The integrand of the surface integral can therefore be interpreted as a force/unit area acting on the

enclosing surface in the ith direction. To distinguish it from a surface force density, it will be referred to as the "traction." It does not act on a physical surface and has physical significance only when integrated over a closed surface. It is simply the force/unit area that must be integrated over the entire surface to find the net force due to the volume force density

$$T_i = T_{ij}n_j; \quad \vec{T} = \vec{\vec{T}}\cdot\vec{n} \tag{5}$$

In vector notation and in terms of the traction $\vec{T}$, Eq. 4 is written as

$$\int_V \vec{F}dV = \oint_S \vec{T}\cdot\vec{n}da \tag{6}$$

Figure 3.9.1 shows the general relationship of the traction and normal vector. The traction can act in an arbitrary direction relative to the surface.

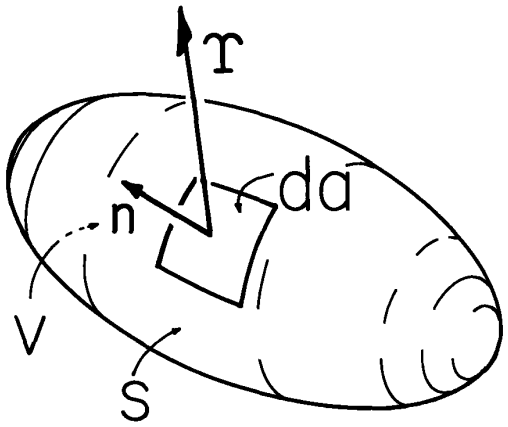

Fig. 3.9.1. Schematic view of volume V enclosed by surface S, showing traction acting on elements of surface.

To develop a physical interpretation of the stress tensor components, it is helpful to consider a particular volume V and surface S with surfaces having normals in the Cartesian coordinate directions. The cube shown in Fig. 3.9.2 is such a volume. Suppose that interest is in determining the net force on the cube in the x direction, from Eq. 4. The required surface integration can then be broken into separate integrations over each of the cube's surfaces. For the integration on the right face, the normal vector has only an x component, so the only contribution to that surface integration is from T_{xx}. Similarly, on the left surface, the normal vector is in the -x direction, and the integral over that surface is of $-T_{xx}$. The minus sign is represented by directing the stress arrow in the minus x direction in Fig. 3.9.2. On the top and bottom surfaces, the normal vector is in the y direction, and the integration is of plus and minus T_{xy}. Similarly, on the front and back surfaces, the only terms contributing to the traction are T_{xz}. The stress tensor components represent normal stresses if the indices are equal, and shear stresses if they are unequal. In either case, the stress component acting in the ith direction on a surface having its normal in the jth direction is T_{ij}.

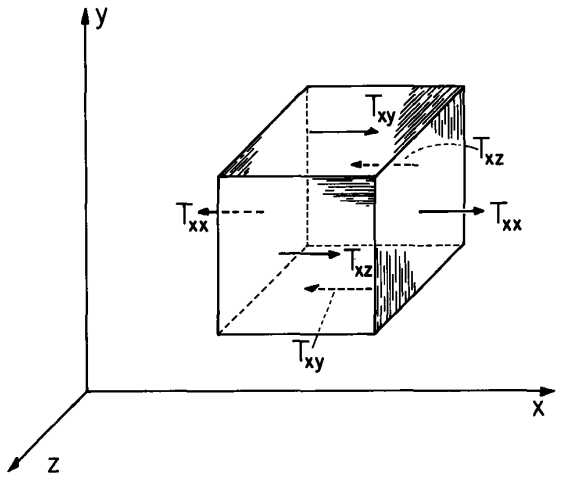

Fig. 3.9.2. Stress components acting on cube in the x direction.

Orthogonal components are a familiar way of representing a vector $\vec{F}$. In the coordinate system (x_1,x_2,x_3) the components are denoted by F_j. What is meant by a vector is implicit to how these components decompose into the components of the vector expressed in a second orthogonal coordinate system $(x_1',x_2'.x_3')$ pictured in Fig. 3.9.3. The two coordinate systems are related by the transformation

$$x_k' = a_{k\ell}x_\ell; \quad \frac{\partial x_k'}{\partial x_\ell} = a_{k\ell} \tag{7}$$

where $a_{k\ell}$ is the cosine of the angle between the x_k' axis and the x_ℓ axis.

A component of the vector in the primed frame in the ith direction is then given by

$$F_i' = a_{ij}F_j \tag{8}$$

For example, suppose that i = 1. Then, Eq. 8 gives the x_1' component of $\vec{F}'$ as the projections of the components in the x_1, x_2, x_3 directions onto the x_1' direction. Equation 8 summarizes how a vector transforms from one coordinate system onto another, and could be used to define what is meant by a "vector."

Similarly, the components of a tensor transform from the unprimed to the primed coordinate system in a way that can be used to define what is meant by a "tensor." To deduce the transformation, begin with Eq. 8 using the divergence of a stress tensor to represent each of the force densities (Eq. 1):

$$\frac{\partial T'_{ik}}{\partial x'_k} = a_{ij} \frac{\partial T_{j\ell}}{\partial x_\ell} \tag{9}$$

Now, if use is made of the chain rule for differentiation, and Eq. 7, it follows that

$$\frac{\partial T'_{ik}}{\partial x'_k} = a_{ij} \frac{\partial T_{j\ell}}{\partial x'_k} \frac{\partial x'_k}{\partial x_\ell} = a_{ij} a_{k\ell} \frac{\partial T_{j\ell}}{\partial x'_k} \tag{10}$$

Thus, the tensor transformation follows as

$$T'_{ik} = a_{ij} a_{k\ell} T_{j\ell} \tag{11}$$

Useful conditions on the direction cosines a_{ij} are obtained by recognizing that the transformation from the primed frame to the unprimed frame, given generally by

$$F_j = b_{ji} F'_i \tag{12}$$

involves the same direction cosines, because b_{ji}, defined as the cosine of the angle between the x_j axis and the x'_i axis, is equal to a_{ij}. Thus, Eqs. 12 and 8 together show that

$$F'_i = a_{ik} F_k = a_{ik} a_{\ell k} F'_\ell \tag{13}$$

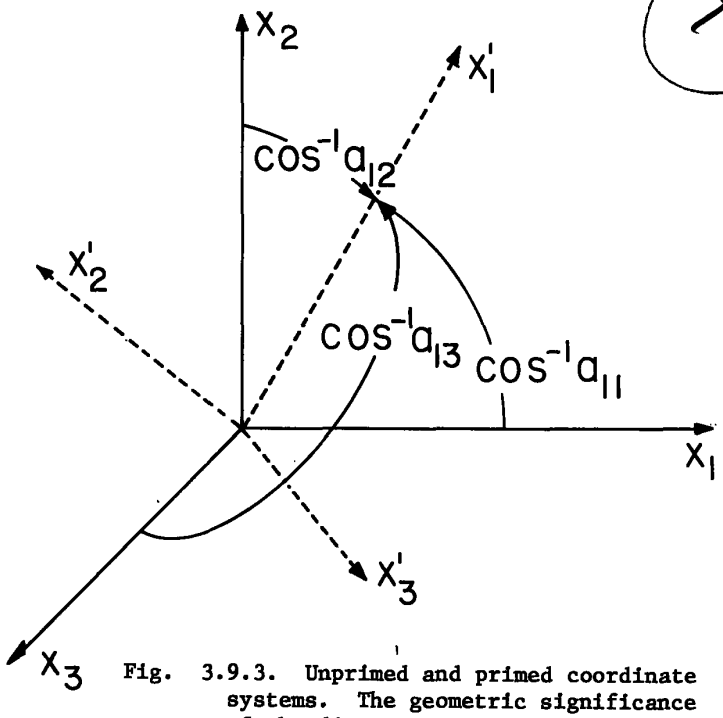

Fig. 3.9.3. Unprimed and primed coordinate systems. The geometric significance of the direction cosine a_{1j} is shown.

and it follows that the direction cosines satisfy the condition that

$$a_{ik} a_{\ell k} = \delta_{i\ell} \tag{14}$$

where the Kronecker delta function δ_{ik} by definition takes the values

$$\delta_{ik} = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases} \tag{15}$$

Finally, suppose that a total torque rather than a total force is to be computed. By way of analogy to Eq. 6, is there a way in which the integration of the torque density can be converted to an integration over the enclosing surface? With respect to the origin, the total torque on material within the volume V is

$$\vec{\tau} = \int_V \vec{r} \times \vec{F} dV \tag{16}$$

where $\vec{r}$ is the vector distance from the origin. With $\vec{F}$ given as the divergence of a stress tensor, Eq. 1, and provided that $\overleftrightarrow{T}$ is symmetric ($T_{ij} = T_{ji}$), the tensor form of Gauss' theorem can be used to show that

$$\vec{\tau} = \oint_S \vec{r} \times (\overleftrightarrow{T} \cdot \vec{n}) da \tag{17}$$

The net torque is the integral over the enclosing surface of a surface torque density $\vec{r} \times \vec{T}$ (see Problem 3.9.1).

3.10 Electromechanical Stress Tensors

The objectives in this section are to illustrate how the stress tensor associated with any one of the force densities in Secs. 3.7 and 3.8 is determined, and to summarize the stress tensors for future reference.

The ith component of the Korteweg-Helmholtz force density, Eq. 3.7.16, written using Gauss' law to eliminate ρ_f, is

$$F_i = E_i \frac{\partial D_j}{\partial x_j} + \sum_{k=1}^{m} \frac{\partial W}{\partial \alpha_k} \frac{\partial \alpha_k}{\partial x_i} - \frac{\partial}{\partial x_i} \left[\sum_{k=1}^{m} \alpha_k \frac{\partial W}{\partial \alpha_k} \right] \tag{1}$$

The goal in the following manipulations is to express this equation in the form of a tensor divergence (in the form of Eq. 3.9.1). The second term can be replaced by Eq. 3.7.26. Also, because $\vec{E}$ is irrotational, $\partial E_i/\partial x_j = \partial E_j/\partial x_i$ and hence Eq. 1 becomes

$$F_i = E_i \frac{\partial D_j}{\partial x_j} + \frac{\partial}{\partial x_i} (W - E_k D_k) + D_j \frac{\partial E_i}{\partial x_j} - \frac{\partial}{\partial x_i} \left[\sum_{k=1}^{m} \alpha_k \frac{\partial W}{\partial \alpha_k} \right] \tag{2}$$

With the first and third terms combined and the Kronecker delta function δ_{ij} introduced (see Eq. 3.9.15),

$$F_i = \frac{\partial}{\partial x_j} \left[E_i D_j + \delta_{ij} \left(W - E_k D_k - \sum_{k=1}^{m} \alpha_k \frac{\partial W}{\partial \alpha_k} \right) \right] \tag{3}$$

It follows from a comparison of Eqs. 2 and 3.9.1 that the required stress tensor is

$$T_{ij} = E_i D_j - \delta_{ij} \left(W' + \sum_{k=1}^{m} \alpha_k \frac{\partial W}{\partial \alpha_k} \right) \tag{4}$$

where the coenergy density, W', is defined by Eq. 2.13.11.

Table 3.10.1 gives a summary of this and other stress tensors together with the associated force densities. It is essential that a consistent pair be used.

Table 3.10.1. Summary of force densities and associated stress tensors.

Equation	Force density	Stress tensors
3.7.16	$\vec{F} = \rho_f \vec{E} + \sum_{k=1}^{m} \frac{\partial W}{\partial \alpha_k} \nabla \alpha_k - \nabla \left[\sum_{k=1}^{m} \alpha_k \frac{\partial W}{\partial \alpha_k} \right]$	$T_{ij} = E_i D_j - \delta_{ij} \left(W' + \sum_{k=1}^{m} \alpha_k \frac{\partial W}{\partial \alpha_k} \right)$
3.8.13	$\vec{F} = \vec{J}_f \times \vec{B} + \sum_{k=1}^{m} \frac{\partial W}{\partial \alpha_k} \nabla \alpha_k - \nabla \left[\sum_{k=1}^{m} \alpha_k \frac{\partial W}{\partial \alpha_k} \right]$	$T_{ij} = H_i B_j - \delta_{ij} \left(W' + \sum_{k=1}^{m} \alpha_k \frac{\partial W}{\partial \alpha_k} \right)$
Incompressible media		
3.7.19	$\vec{F} = \rho_f \vec{E} + \sum_{k=1}^{m} \frac{\partial W}{\partial \alpha_k} \nabla \alpha_k$	$T_{ij} = E_i D_j - \delta_{ij} W'$
3.8.14	$\vec{F} = \vec{J}_f \times \vec{B} + \sum_{k=1}^{m} \frac{\partial W}{\partial \alpha_k} \nabla \alpha_k$	$T_{ij} = H_i B_j - \delta_{ij} W'$
Incompressible and electrically linear: $\vec{D} = \varepsilon \vec{E}, \vec{B} = \mu \vec{H}$		
3.7.22	$\vec{F} = \rho_f \vec{E} - \frac{1}{2} E^2 \nabla \varepsilon$	$T_{ij} = \varepsilon E_i E_j - \frac{\varepsilon}{2} \delta_{ij} E_k E_k$
3.8.14	$\vec{F} = \vec{J}_f \times \vec{B} - \frac{1}{2} H^2 \nabla \mu$	$T_{ij} = \mu H_i H_j - \frac{\mu}{2} \delta_{ij} H_k H_k$
Electrically linear, ε and μ dependent on mass density ρ only		
3.7.24	$\vec{F} = \rho_f \vec{E} - \frac{1}{2} E^2 \nabla \varepsilon + \nabla \left(\frac{1}{2} \rho \frac{\partial \varepsilon}{\partial \rho} E^2 \right)$	$T_{ij} = \varepsilon E_i E_j - \frac{\varepsilon}{2} \delta_{ij} E_k E_k \left(1 - \frac{\rho}{\varepsilon} \frac{\partial \varepsilon}{\partial \rho} \right)$
3.8.17	$\vec{F} = \vec{J}_f \times \vec{B} - \frac{1}{2} H^2 \nabla \mu + \nabla \left(\frac{1}{2} \rho \frac{\partial \mu}{\partial \rho} H^2 \right)$	$T_{ij} = \mu H_i H_j - \frac{\mu}{2} \delta_{ij} H_k H_k \left(1 - \frac{\rho}{\mu} \frac{\partial \mu}{\partial \rho} \right)$
Kelvin force density and stress tensor		
3.6.5	$\vec{F} = \rho_f \vec{E} + \vec{P} \cdot \nabla \vec{E}$	$T_{ij} = E_i D_j - \frac{1}{2} \delta_{ij} \varepsilon_o E_k E_k$
3.5.12	$\vec{F} = \vec{J}_f \times \mu_o \vec{H} + \mu_o \vec{M} \cdot \nabla \vec{H}$	$T_{ij} = H_i B_j - \frac{1}{2} \delta_{ij} \mu_o H_k H_k$

The stress tensor makes it possible to compute the total force on an object by integrating over an enclosing surface S in accordance with Eq. 3.9.6. For an isolated object in free space, this force is the same regardless of the particular force density used. If the force is considered as the integral of the force density over the volume of the object, this fact is by no means obvious. But, note that in free space the stress tensors of Table 3.10.1 all agree. Because the enclosing surface S is in this free space region, the same total force will result from integrating Eq. 3.9.6 regardless of the force density associated with the stress tensor.

3.11 Surface Force Density

In many systems, the electric or magnetic force density is concentrated in a thin layer, usually comprising the interface between two regions. If the thickness of this layer is small compared to the dimensions of the adjacent regions and other lengths of interest, then the force per unit area on the interface may be used to describe the layer. An interfacial section is enclosed by the incremental volume of thickness Δ and area $A = \delta x \delta y$, shown in Fig. 3.11.1. The surface force density is defined as a force per unit area of the interface in a limit in which first Δ and then A approach zero. The integration of the electric force density throughout the control volume is conveniently carried out using the appropriate stress tensor T_{ij} integrated over the enclosing surface. With $\vec{n}$ defined as the unit normal to the interface and $\vec{i}_n$ the unit normal to the control surface, the surface force density is

$$\vec{T} = \lim_{\substack{\Delta\to 0 \\ A\to 0}} \frac{1}{A} \oint_S \vec{\vec{T}}\cdot\vec{i}_n da = [\![\vec{\vec{T}}]\!]\cdot\vec{n} + \lim_{A\to 0} \frac{1}{A} \oint_C \int_{0^-}^{0^+} \vec{\vec{T}}\cdot\vec{i}_n d\nu d\ell \tag{1}$$

Integration is divided into two parts. The first is the contribution from the surfaces external to the layer, having normals $\vec{n}$ and $-\vec{n}$, respectively. The second accounts for the "edges" of the volume where the surface cuts through the double layer. If fields within the layer are of the same order as those outside, contributions of the second integral vanish as $\Delta \to 0$. In electroquasistatic systems, the double layer presents a case where the internal fields are sufficiently intense that the second term not only makes a contribution but one that can dominate the first term. The remainder of this section is devoted to converting this contribution to a more useful form.

The distance normal to the interface is y, with (μ,ξ) orthogonal coordinates in the local interfacial plane, as shown in Fig. 3.11.1. In the absence of a double layer, the electric field is of the same order of magnitude throughout, and hence in the limit $\Delta \to 0$, the second term in Eq. 1 becomes negligible compared to the first. With the double layer, the stress contributions from the edges of the control volume are of the same order as those from the exterior surfaces.

As discussed in Sec. 2.10, the tangential electric field suffers a discontinuity through the double layer. However, the tangential field within the layer is of the same order as the external field. Because the thickness Δ over which the interior stresses act is much smaller than the linear dimensions $\delta\xi$ and $\delta\mu$, the internal stress contributions to the integrations around the periphery of the control volume are ignorable unless the double-layer charges are themselves responsible for a substantially larger internal field than external field. This double-layer-generated field is directed normal to the interface and dominates in determining the interior stresses. The stress taken now as represented by Eq. 3.7.19b of Table 3.10.1 is

$$T_{ij} = E_i D_j - \delta_{ij} W' \tag{2}$$

where, in the case of a linearly polarized dielectric, the coenergy density W' is simply $\varepsilon E^2/2$. Stress components associated with the dominant field in the double layer interior are essentially

$$\begin{aligned} &T_{\xi\xi} \to T_{\mu\mu} \to -W' \\ &T_{ij} \to 0; \quad i \neq j \end{aligned} \tag{3}$$

The traction acting on the periphery of the control volume is therefore approximately

$$\int_{0^-}^{0^+} \mathbf{T}\cdot\vec{i}_n d\nu = -\int_{0^-}^{0^+} W' d\nu \vec{i}_n \equiv \gamma_E \vec{i}_n \tag{4}$$

The normal vector $\vec{i}_n$ can be written as $-\vec{n}\times d\vec{\ell}$, so that Eq. 1 becomes

$$\vec{T} = [\![\vec{\vec{T}}]\!]\cdot\vec{n} - \lim_{A\to 0} \frac{1}{A} \oint_C \gamma_E \vec{n}\times d\vec{\ell}$$

In the limit $A \to 0$, the contour integral in Eq. 5 need only be evaluated to first order in $\delta\xi\cdot\delta\mu$. Expansion about the origin, denoted by the subscript o, gives an approximate expression for the integral

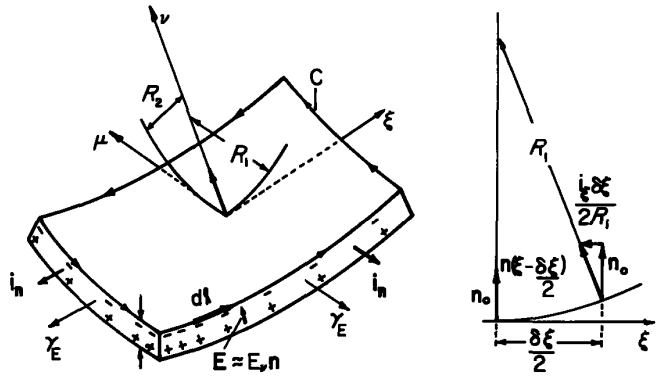

Fig. 3.11.1

(a) Volume enclosing section of interface. Thickness Δ is sufficient to include double layer but small compared to linear dimensions of A. (b) Cross-sectional view of interface showing relation of radius of curvature R to n and dℓ.

that becomes exact in the limit. The contour C is taken as rectangular with edges parallel to the (ξ,μ) axes. The segment of length $\delta\mu$ at $\xi = \delta\xi/2$ has $-\vec{n}\times d\vec{\ell} \approx \delta\mu(\vec{i}_\xi + \vec{n}_o \delta\xi/R_1)$ and gives a contribution to the contour integral

$$\left\{[\gamma_E]_o + [\frac{\partial\gamma_E}{\partial\xi}]_o \frac{\delta\xi}{2}\right\}\left\{\vec{i}_\xi + \frac{\vec{n}_o \delta\xi}{R_1}\right\}\delta\mu \tag{6}$$

The three additional sides of the rectangular contour give similar contributions, so that altogether,

$$\begin{aligned}
-\lim_{A\to 0} \frac{1}{A}\oint_C \gamma_E \vec{n}\times d\vec{\ell} = \lim_{\delta\xi\delta\mu\to 0} \frac{1}{\delta\xi\delta\mu} \Bigg| &\left\{[\gamma_E]_o + [\frac{\partial\gamma_E}{\partial\xi}]_o \frac{\delta\xi}{2}\right\}\left\{\vec{i}_\xi + \frac{\vec{n}_o}{R_1}\frac{\delta\xi}{2}\right\}\delta\mu \\
&+\left\{[\gamma_E]_o - [\frac{\partial\gamma_E}{\partial\xi}]_o \frac{\delta\xi}{2}\right\}\left\{-\vec{i}_\xi + \frac{\vec{n}_o}{R_1}\frac{\delta\xi}{2}\right\}\delta\mu + \left\{[\gamma_E]_o + [\frac{\partial\gamma_E}{\partial\mu}]_o \frac{\delta\mu}{2}\right\}\left\{\vec{i}_\mu + \frac{\vec{n}_o}{R_2}\frac{\delta\mu}{2}\right\}\delta\xi \\
&+\left\{[\gamma_E]_o - [\frac{\partial\gamma_E}{\partial\mu}]_o \frac{\delta\mu}{2}\right\}\left\{-\vec{i}_\mu + \frac{\vec{n}_o}{R_2}\frac{\delta\mu}{2}\right\}\delta\xi\Bigg| \\
&= \vec{n}\gamma_E[\frac{1}{R_1} + \frac{1}{R_2}] + \nabla_\Sigma\gamma_E
\end{aligned} \tag{7}$$

Here, R_1 and R_2 are radii of curvature for the interface, reckoned in the orthogonal planes defined respectively by the normal and ξ and the normal and μ. Note that the sign of each curvature term is taken as positive if the center of curvature is on the side of the interface toward which $\vec{n}$ is directed. The surface force density associated with surface tension takes this same form. However, the convention used in Chap. 7 is with the radii of curvature the negatives of R_1 and R_2. With the understanding that R_1 and R_2 are radii of curvature taken as positive if the center of curvature is on the side of the interface out of which $\vec{n}$ is directed, Eqs. 1, 4, and 7 give the surface force density, with the double-layer contribution represented by the function γ_E,

$$\vec{T} = [\![\, \vec{\vec{T}} \,]\!]\cdot\vec{n} - \vec{n}\gamma_E[\frac{1}{R_1} + \frac{1}{R_2}] + \nabla_\Sigma\gamma_E \tag{8}$$

where

$$\gamma_E \equiv \int_{0^-}^{0^+} W' d\nu$$

It is shown in Sec. 7.6 that the second term in Eq. 8 can also be expressed as $-\gamma_E(\nabla\cdot\vec{n})\vec{n}$.

The double layer surface force density is exemplified in Chap. 10.

3.12 Observations

The force densities and associated stress tensors of Table 3.10.1 are of two origins. The Kelvin force densities, the last two in the table, come from a microscopic picture of particles and dipoles subject to electric or magnetic forces which, through the agent of a kinetic equilibrium, are passed along to the ponderable continuum. The Korteweg-Helmholz force densities, all of the others in the table, are based on an energy conservation principle. The connection between micro and macro fields, needed to apply this principle, is made using electrical measurements of constitutive laws to interrelate the macroscopic fields $\vec{D}$ and $\vec{E}$ or $\vec{B}$ and $\vec{H}$.

The arguments underlying each type of force density envoke certain assumptions which point to possible inadequacies. The Kelvin force densities picture the force acting on each dipole and each point charge in isolation and this force as being that transmitted to the ponderable media. This does not allow for the possibility that the micro fields of one dipole contribute to the force on a neighboring dipole.

This shortcoming is obviated by the energy method, which is based on a statement of energy conservation for an electromechanical subsystem. The resulting Korteweg-Helmholtz force densities [1] are of course also restricted. On the one hand, they are more broadly applicable than might be concluded from the derivations. For example, the MQS continuum is viewed as "perfectly conducting," but the free current force density is certainly applicable in cases where the conductivity is finite. This is evident from its agreement with the Lorentz force density of Sec. 3.1, because the later model includes a finite mobility and hence electrical dissipation.

One way to derive a force density without ambiguity as to the validity of the result in nonconservative systems is to replace statements of energy conservation with those of power flow.[2] However, the principle of virtual power requires information beyond that required by the principle of virtual work used here. In addition to the constitutive laws relating the macroscopic field variables is the requirement for the power flux density, which must either be assumed or measured.

Underlying all of the discussions in this chapter has been the presumption that a clear distinction can be made between electric or magnetic force densities and those of other origins. This is tantamount to being able to isolate electromagnetic energy storage from other forms of energy storage. Piezoelectric coupling is an example where it is not fruitful to make this distinction. In that area, the stress and force density generally represent combined electric and mechanical electromechanical effects.

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1. J. A. Stratton, Electromagnetic Theory, McGraw-Hill Book Co. Inc., New York, 1941, pp. 137-159.
2. P. Penfield, Jr., and H. H. Haus, Electrodynamics of Moving Media, The M.I.T. Press, Cambridge, Massachusetts, 1967, pp. 35-40.