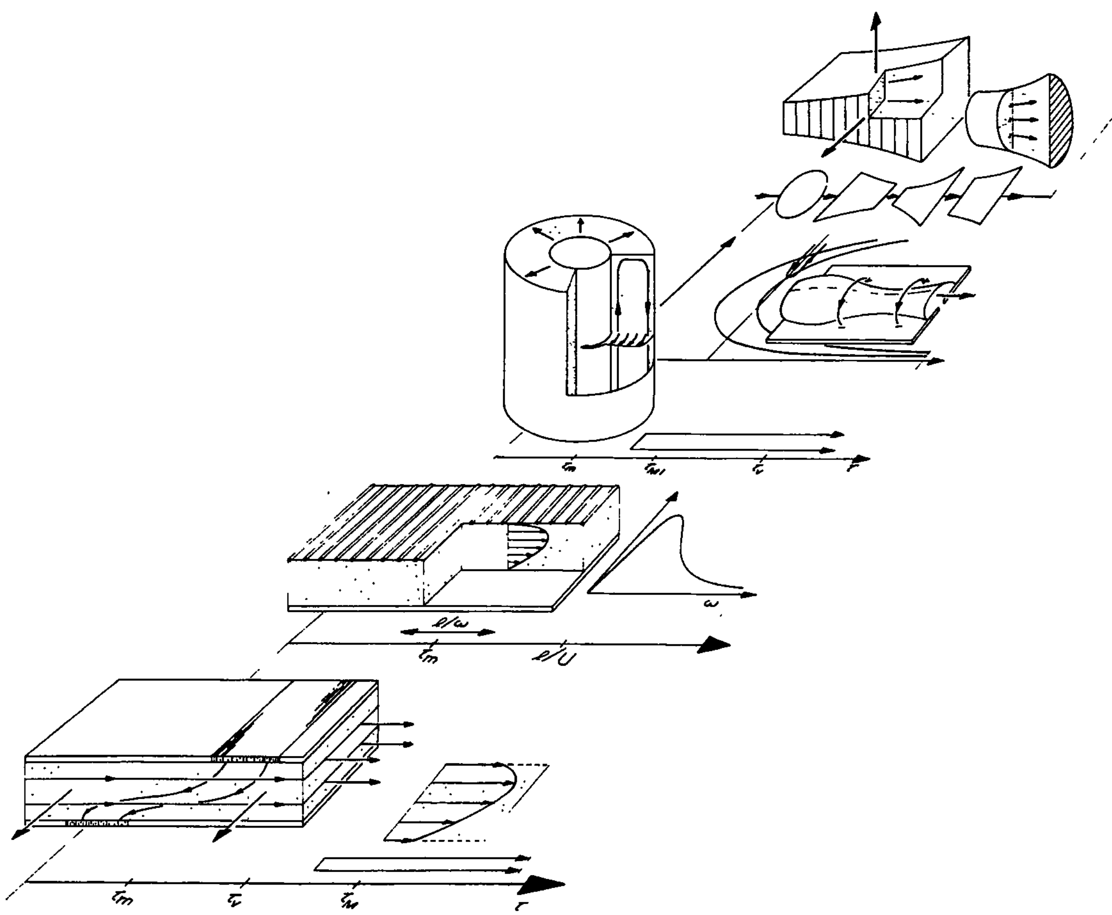
Solutions Manual for Continuum Electromechanics

9

Electromechanical Flows



Prob. 9.3.1 (a) With $\partial p'/\partial y = 0$ and $T_{xy}=0$, Eq. (a) reduces to $v = v^\beta + (v^\alpha - v^\beta)(\frac{x}{\Delta})$ Thus, the velocity profile is seen to be linear in x. (b) With $v^\beta = v^\alpha = 0$ and $T_{yx}=0$, Eq. (a) becomes

$$v(x) = \frac{\Delta^2}{2\eta}\frac{\partial p'}{\partial y}\left[\left(\frac{x}{\Delta}\right) - 1\right]\frac{x}{\Delta}$$

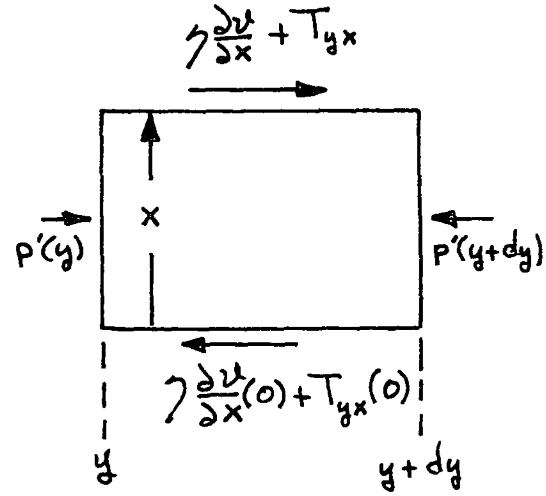
and the velocity profile is seen to be parabolic. The peak velocity is at the center of the channel, where it is $-(\Delta^2/8\eta)\partial p'/\partial y$. The volume rate of flow follows as

$$Q_v = w\Delta\int_0^1 v\,d\left(\frac{x}{\Delta}\right) = \frac{w\Delta^3}{2\eta}\frac{\partial p'}{\partial y}\left[\frac{1}{3}\left(\frac{x}{\Delta}\right)^3 - \frac{1}{2}\left(\frac{x}{\Delta}\right)^2\right]_0^1 = \frac{-w\Delta^3}{12\eta}\frac{\partial p'}{\partial y}$$

Hence, the desired relation of volume rate of flow and the difference between outlet pressure and inlet pressure, Δp, is

$$Q_v = \frac{-w\Delta^3}{12\eta}\left(\frac{\Delta p}{\ell}\right)$$

Prob. 9.3.2 The control volume is as shown with hybrid pressure p' acting on the longitudinal surfaces (which have height x) and shear stresses acting on transverse surface. With the assumption that these surface stresses represent all of the forces (that there is no acceleration), the force equilibrium is represented by



$$[p'(y+dy) - p'(y)]x = \left(\eta\frac{\partial v}{\partial x} + T_{yx}\right)dy - \left(\eta\frac{\partial v}{\partial x}(0) + T_{yx}(0)\right)dy$$

Divided by dy, this expression becomes Eq. (5)

Prob. 9.3.3 Unlike the other fully developed flows in Table 9.3.1, this one involves an acceleration. The Navier-Stokes equation is

$$(\bar{v}\cdot\nabla)\bar{v} + \nabla p = \nabla(\rho\bar{g}\cdot\bar{r}) + \eta\nabla^2\bar{v} + \nabla\cdot\bar{\bar{T}} \tag{1}$$

With $\bar{v} = v(r)\,\bar{i}_\theta$, continuity is automatically satisfied, $\nabla\cdot\bar{v} = 0$. The radial component of Eq. 1 is

$$-\frac{v^2}{r} + \frac{\partial p}{\partial r} = \frac{\partial}{\partial r}(\rho\bar{g}\cdot\bar{r}) + F_r(r) \tag{2}$$

It is always possible to find a scalar $\mathcal{E}(r)$ such that $F_r = -\partial\mathcal{E}/\partial r$ and to define a scalar $T(r)$ such that $T = -\int(v^2/r)\,dr$. Then, Eq. (2) reduces to

$$\frac{\partial p'}{\partial r} = 0\;;\; p' \equiv p + T(r) + \mathcal{E}(r) - \rho\bar{g}\cdot\bar{r} \tag{3}$$

The θ component of Eq. (1) is best written so that the viscous shear stress is evident. Thus, the viscous term is written as the divergence of the viscous stress tensor, so that the θ component of Eq. (1) becomes

$$\frac{1}{r}\frac{\partial p'}{\partial\theta} = \frac{\partial}{\partial r}\left(T_{r\theta} + T^{v}_{r\theta}\right) + \frac{2}{r}\left(T_{r\theta} + T^{v}_{r\theta}\right) \tag{4}$$

where

$$T^{v}_{r\theta} = \eta r\frac{\partial}{\partial r}\left(\frac{v}{r}\right) \tag{5}$$

Multiplication of Eq. (4) by r^2 makes it possible to write the right hand side as a perfect differential.

$$r\frac{\partial p'}{\partial\theta} = \frac{\partial}{\partial r}\left[r^2\left(T_{r\theta} + T^{v}_{r\theta}\right)\right] \tag{6}$$

Then, because the flow is reentrant, $\partial p'/\partial\theta = 0$ and Eq. (6) can be integrated.

$$r^2\left[T_{r\theta} + \eta r\frac{d}{dr}\left(\frac{v}{r}\right)\right] = C \tag{7}$$

a second integration of Eq. (7) divided by r^3 gives

$$\int_\beta^r \frac{T_{r\theta}}{r}\,dr + \eta\left(\frac{v}{r} - \frac{v^\beta}{\beta}\right) = \int_\beta^r \frac{C}{r^3}\,dr = -\frac{C}{2}\left(\frac{1}{r^2} - \frac{1}{\beta^2}\right) \tag{8}$$

Prob. 9.3.3 (cont.)

The coefficient C is determined in terms of the velocity v^α on the outer surface by evaluating Eq. (8) on the outer boundary and solving for C.

$$C = \frac{2}{\left(\frac{1}{\beta^2} - \frac{1}{\alpha^2}\right)}\left[\int_\beta^\alpha \frac{T_{r\theta}}{r}\,dr + \eta\left(\frac{v^\alpha}{\alpha} - \frac{v^\beta}{\beta}\right)\right] \tag{9}$$

This can now be introduced into Eq. (8) to give the desired velocity distribution, Eq. (b) of Table 9.2.1.

Prob. 9.3.4 With $T_{r\theta}=0$, Eq. (b) of Table 9.2.1 becomes

$$v = \frac{1}{\left(\frac{\alpha}{\beta} - \frac{\beta}{\alpha}\right)}\left[v^\alpha\left(\frac{r}{\beta} - \frac{\beta}{r}\right) + v^\beta\left(\frac{\alpha}{r} - \frac{r}{\alpha}\right)\right] \tag{1}$$

The viscous stress follows as

$$T^v_{r\theta} = \eta r \frac{\partial}{\partial r}\left(\frac{v}{r}\right) = \frac{\eta r}{\left(\frac{\alpha}{\beta} - \frac{\beta}{\alpha}\right)}\left(\frac{2v^\alpha\beta}{r^3} - \frac{2v^\beta\alpha}{r^3}\right) \tag{2}$$

Substituting $v^\alpha = \alpha\Omega_a$ and $v^\beta = \beta\Omega_b$, at the inner surface where $r = \beta$ this becomes

$$T^v_{r\theta} = \frac{-2\eta\alpha}{\beta\left(\frac{\alpha}{\beta} - \frac{\beta}{\alpha}\right)}(\Omega_b - \Omega_a) \tag{3}$$

The torque on the inner cylinder is its area multiplied by the lever-arm β and the stress $T_{r\theta}$.

$$T = (2\pi W\beta)\beta\left(T^v_{r\theta}\right)^\beta = \frac{-4\pi\beta^2 W\eta\alpha}{\beta\left(\frac{\alpha}{\beta} - \frac{\beta}{\alpha}\right)}(\Omega_b - \Omega_a) \tag{4}$$

Note that in the limit where the outer cylinder is far away, this becomes

$$T = -4\pi\beta^2 W\eta(\Omega_b - \Omega_a) \tag{5}$$

(b) Expand the term multiplying v^α in Eq. (1) letting $r = \beta + r'$, $r' \ll \beta$ so that $r^{-1} \cong (1/\beta - r'/\beta^2)$. In the term multiplying v^β, expand $r = \alpha - r''$ so that $r^{-1} = (1/\alpha + r''/\alpha^2)$. Thus, Eq. (1) becomes

$$v \simeq \frac{\alpha\beta}{(\alpha-\beta)(\alpha+\beta)}\left[v^\alpha\frac{2r'}{\beta} + v^\beta\frac{2r''}{\alpha}\right] \tag{6}$$

Prob. 9.3.4(cont.)

The term out in front becomes approximately $\alpha/(\alpha-\beta)2$.

Thus, with the identification $r' \to x$, $r'' \to \Delta - x$ and $\alpha-\beta \to \Delta$ the velocity profile becomes

$$v = v^{\alpha}\left(\frac{x}{\Delta}\right) + v^{\beta}\left(1 - \frac{x}{\Delta}\right) \tag{7}$$

which is the plane Couette flow profile (Prob. 9.2.1).

<u>Prob. 9.3.5</u> With the assumption $\bar{v} = v(r)\bar{i}_z$, continuity is automatically satisfied and the radial component of the Navier Stokes equation becomes

$$\frac{\partial p}{\partial r} = \frac{\partial}{\partial r}(\rho \bar{g}\cdot\bar{r}) + F_r(r) \;;\; F_r = -\frac{d\mathcal{E}}{dr} \tag{1}$$

so that the radial force density is balanced by the pressure in such a way that p' is independent of r, where $p' \equiv p - \rho\bar{g}\cdot\bar{r} + \mathcal{E}$.

Multiplied by r, the longitudinal component of the Navier Stokes equation is

$$r\frac{\partial p'}{\partial z} = \frac{\partial}{\partial r}(rT_{zr}) + \eta\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) \tag{2}$$

This expression is integrated to obtain

$$\frac{1}{2}\frac{\partial p'}{\partial z}(r^2-\beta^2) = rT_{zr} - \beta T_{zr}^{\beta} + \eta\left[r\frac{\partial v}{\partial r} - \beta\left(\frac{\partial v}{\partial r}\right)^{\beta}\right] \tag{3}$$

A second integration of this expression multiplied by r leads to the velocity v(r)

$$\frac{1}{2}\frac{\partial p'}{\partial z}\left[\frac{1}{2}(r^2-\beta^2) - \beta^2 \ln\left(\frac{r}{\beta}\right)\right] = \int_{\beta}^{r} T_{zr}\,dr - \beta T_{zr}^{\beta}\ln\left(\frac{r}{\beta}\right) + \eta(v - v^{\beta}) - \eta\beta\left(\frac{\partial v}{\partial r}\right)^{\beta}\ln\left(\frac{r}{\beta}\right)$$

in terms of the constant $(\partial v/\partial r)^{\beta}$. To replace this constant with the velocity evaluated on the outer boundary, Eq. (4) is evaluated at the outer boundary, r = α, where $v = v^{\alpha}$ and that expression solved for $(\partial v/\partial r)^{\beta}$. Substitution of the resulting expression into Eq. (4) gives an expression that can be solved for the velocity profile in terms of v^{α} and v^{β}, Eq. (c) of Table 9.2.1.

Prob. 9.3.6 This problem is probably more easily solved directly than by taking the limit of Eq. (c). However, it is instructive to take the limit. Note that $T_{zr}=0$, $v^{\alpha}=0$ and $\alpha=R$. But, so long as v^{β} is finite, the term $\beta^2 \ln(r/\beta)/\ln(\alpha/\beta)$ goes to zero as $\beta \to 0$. Moreover,

$$\lim_{\beta\to 0} \alpha^2 \ln(r/\beta)/\ln(\alpha/\beta) = \lim_{\beta\to 0} \alpha^2 \frac{[\ln(r)-\ln(\beta)]}{[\ln(\alpha)-\ln(\beta)]} = \alpha^2$$

so that the required circular Couette flow has a parabola as its profile

$$v = \frac{1}{4\eta}\frac{\partial p'}{\partial z}(r^2 - R^2)$$

(b) The volume rate of flow follows from Eq. (2)

$$Q_v = \int_0^R v\, 2\pi r\, dr = -\frac{\pi}{8\eta} R^4 \frac{dp'}{dz} = -\frac{\pi}{8\eta} R^4 \frac{\Delta p}{\lambda}$$

where Δp is the pressure at the outlet minus that at the inlet.

Prob. 9.4.1 Equation 5.14.11 gives the surface force density in the form

$$\langle T_z \rangle_z = c(\epsilon_a \sigma_b - \epsilon_b \sigma_a)\frac{S_E}{1 + S_E^2} \equiv T_o \tag{1}$$

Thus, the interface tends to move in the positive y direction if the upper region (the one nearest the electrode) is insulating and the lower one is filled with semi-insulating liquid and if S_E is greater than zero, which it is if the wave travels in the y direction and the interface moves at a phase velocity less than that of the wave.

For purposes of the fluid mechanics analysis, the coordinate origin for x is moved to the bottom of the tank. Then, Eq. (a) of Table 9.3.1 is applicable with $v^{\beta}=0$ and $v^{\alpha} = U$ (the unknown surface velocity). There are no internal force densities in the y direction, so $T_{yx}=0$. In this expression, there are two unknowns, U and $\partial p'/\partial y$. These are determined

Prob. 9.4.1 (cont.)

by the stress balance at the interface, which requires that

$$\eta \frac{\partial v_y}{\partial x}\bigg|_{x=0} = T_o \tag{2}$$

and the condition that mass be conserved.

$$\int_0^b v_y \, dx = 0 \tag{3}$$

These require that

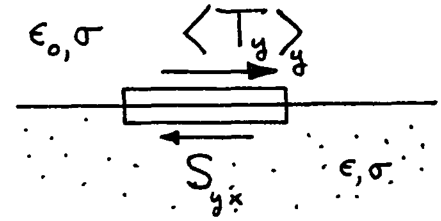
$$\begin{bmatrix} \frac{\eta}{b} & \frac{b}{2} \\ \frac{b}{2} & -\frac{b^3}{12\eta} \end{bmatrix} \begin{bmatrix} U \\ \frac{\partial p'}{\partial y} \end{bmatrix} = \begin{bmatrix} T_o \\ 0 \end{bmatrix} \tag{4}$$

and it follows that $U = bT_o/4\eta$ and $\partial p'/\partial y = 3T_o/2b$

so that the required velocity profile, Eq. (a) of Table 9.3.1 is

$$v_y = \frac{bT_o}{4\eta} \frac{x}{b}\left(3\frac{x}{b} - 2\right) \tag{5}$$

<u>Prob. 9.4.2</u> The time average electric surface force density is found by adapting Eq. 5.14.11. That configuration models the upper region and the infinite half space if it is turned upside down and $z \to y$, $a \to a$, $\epsilon_a \to \epsilon$, $\epsilon_b \to \epsilon_o$, $\sigma_a \to \sigma$, $\sigma_b \to 0$ and $b \to \infty$. Then,



$$\langle T_y \rangle_y = -\frac{1}{2}\epsilon |k\hat{V}_o|^2 K \epsilon_o \sigma \frac{S_E}{1 + S_E^2} \tag{1}$$

where

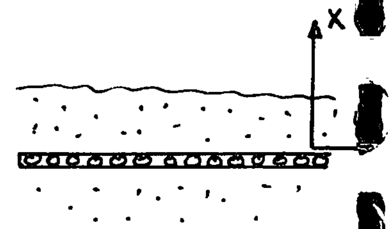
$$S_E \equiv \omega \tau_E \left(1 - \frac{kU}{\omega}\right)$$

$$\tau_E \equiv \frac{\epsilon \coth ka + \epsilon_o}{\sigma \coth ka} \qquad (k > 0)$$

$$K = \left\{ \sinh^2 ka \left[\epsilon \coth ka + \epsilon_o\right]\left[\sigma \coth ka\right]\right\}^{-1}$$

Note that for $|\omega| > |kU|$, the electric surface force density is negative.

Prob. 9.4.2 (cont.)



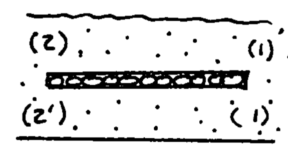
With x defined as shown to the right, Eq. (a) of Table 9.3.1 is adapted to the flow in the upper section by setting $v^\alpha = U$, $v^\beta = 0$, $\Delta \to a$ and $T_{yx} = 0$ so that

$$v(x) = U\frac{x}{a} + \frac{a^2}{2\eta}\frac{\partial p'}{\partial y}\left[\left(\frac{x}{a}\right)^2 - \frac{x}{a}\right] \tag{2}$$

From this, the viscous shear stress follows as

$$S_{yx} = \eta\frac{\partial v}{\partial x} = \frac{\eta U}{a} + \frac{a}{2}\frac{\partial p'}{\partial y}\left(\frac{2x}{a} - 1\right) \tag{3}$$

Thus, shear stress equilibrium at the interface requires that $(\partial p'/\partial y \equiv (p^1 - p^2)/\ell)$



$$\langle T_y \rangle_y = S_{yx}(x=a) = \frac{\eta U}{a} + \frac{a}{2}\frac{(p^1 - p^2)}{\ell} \tag{4}$$

Thus,

$$U = \frac{a}{\eta}\langle T_y \rangle_y - \frac{a^2}{2\eta}\frac{(p^1 - p^2)}{\ell} \tag{5}$$

and Eq. 2 becomes

$$v(x) = \left[\frac{a}{\eta}\langle T_y \rangle_y - \frac{a^2}{2\eta}\frac{(p^1 - p^2)}{\ell}\right]\frac{x}{a} + \frac{a^2}{2\eta}\frac{p^1 - p^2}{\ell}\left[\left(\frac{x}{a}\right)^2 - \frac{x}{a}\right] \tag{6}$$

It is the volume rate of flow that is in common to the upper and lower regions. For the upper region

$$Q_v \equiv \int_0^a v(x)\,dx = \frac{a^2}{2\eta}\langle T_y \rangle_y - \frac{(p^1 - p^2)}{\ell}\frac{a^3}{3\eta} \tag{7}$$

In the lower region, where $\Delta \to b$, $v^\beta = v^\alpha = 0$ $\quad T_{yx} = 0$, Eq. (a) of Table 9.3.1 becomes ($p^1 \approx p^{1'}$ and $p^2 \approx p^{2'}$)

$$v = \frac{b^2}{2\eta}\frac{(p^1 - p^2)}{\ell}\left[\left(\frac{x}{b}\right)^2 - \frac{x}{b}\right] \tag{8}$$

Thus, in the lower region, the volume rate of flow is

Prob. 9.4.2 (cont.)

$$Q_v = \int_0^b v(x)dx = -\frac{b^3}{12\eta}\frac{(p^1-p^2)}{\ell} \tag{9}$$

Because Q_v in the upper and lower sections must sum to zero, it follows from Eqs. 7 and 9 that

$$\frac{p^1-p^2}{\ell} = \frac{\frac{a^2}{2\eta}\langle T_y\rangle_y}{\frac{a^3}{3\eta}+\frac{b^3}{12\eta}} = \frac{6a^2\langle T_y\rangle_y}{4a^3+b^3} \tag{10}$$

This expression is then substituted into Eq. 5 to obtain the surface velocity, U.

$$U = \frac{a}{\eta}\left[\frac{3a^3+b^3}{4a^3+b^3}\right]\langle T_y\rangle_y \tag{11}$$

Note that because $\langle T_y\rangle_y$ is negative (if the imposed traveling wave of potential travels to the right with a velocity greater than that of the fluid in that same direction) the actual velocity of the interface is to the left, as illustrated in Fig. 9.4.2b.

Prob. 9.4.3 It is assumed that the magnetic skin depth is very short compared to the depth b of the liquid. Thus, it is appropriate to model the electromechanical coupling by a surface force density acting at the interface of the liquid. First, what is the magnetic field distribution under the assumption that $v_y \ll \omega/k$, so that there is no effect of the liquid motion on the field? In the air gap, Eqs. (a) of Table 6.5.1 with σ =0 show that

$$\begin{bmatrix}\hat{H}_x^a\\ \hat{H}_x^b\end{bmatrix} = -j\begin{bmatrix}-\coth ka & \frac{1}{\sinh ka}\\ \frac{-1}{\sinh ka} & \coth ka\end{bmatrix}\begin{bmatrix}\hat{H}_y^a\\ \hat{H}_y^b\end{bmatrix} \tag{1}$$

while in the liquid, Eq. 6.8.5 becomes

$$\hat{H}_x^c = \frac{1}{2}(1+j)k\delta\,\hat{H}_y^c \tag{2}$$

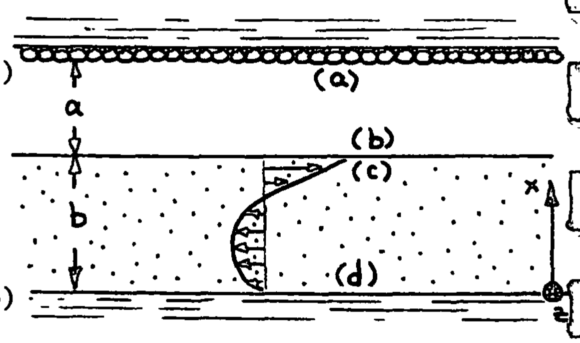
Boundary conditions are

$$\hat{H}_y^a = -\hat{K}_o,\ \mu_o\hat{H}_x^b = \mu\hat{H}_x^c,\ \hat{H}_y^b = \hat{H}_y^c \tag{3}$$

Prob. 9.4.3 (cont.)

Thus, it follows that

$$\hat{H}_y^b = -j\hat{K}_o \Big/ \left\{ \sinh ka \left[\frac{1}{2} k\delta \frac{\mu}{\mu_o} + j\left(\coth ka + \frac{1}{2} k\delta \frac{\mu}{\mu_o} \right) \right] \right\} \quad (4)$$



It follows from Eq. 6.8.10 that the time-average surface force density is

$$\langle T_y \rangle_t = \frac{1}{4} \mu k \delta |\hat{H}_y^b|^2 \quad (5)$$

Under the assumption that the interface remains flat, shear stress balance at the interface requires that

$$\eta \left[\frac{\partial v_y}{\partial x} \right]^{(c)} = \frac{1}{4} \mu k \delta \frac{|\hat{K}_o|^2}{\sinh^2 ka \left\{ \left(\frac{1}{2} k\delta \frac{\mu}{\mu_o} \right)^2 + \left(\coth ka + \frac{1}{2} k\delta \frac{\mu}{\mu_o} \right)^2 \right\}} \quad (6)$$

The fully developed flow, Eq. (a) from Table 9.3.1, is used with the bulk shear stress set equal to zero and $v^d=0$. That there is no net volume rate of flow is represented by

$$v_y^c = \frac{b^2}{6\eta} \frac{\partial p}{\partial y} \quad (7)$$

So, in terms of the "to be determined" surface velocity, the profile is

$$v_y = 3\left(\frac{x}{b} - \frac{2}{3} \right)\left(\frac{x}{b} \right) v^c \quad (8)$$

The surface velocity can now be determined by using this expression to evaluate the shear stress balance of Eq. 6.

$$\eta \left[\frac{\partial v_y}{\partial x} \right]^c = \frac{4\eta}{b} v^c \quad (9)$$

Thus, the required surface velocity is

$$v^c = \frac{\mu k \delta b}{16\eta} \frac{|\hat{K}_o|^2}{\sinh^2 ka \left\{ \left(\frac{1}{2} k\delta \frac{\mu}{\mu_o} \right)^2 + \left(\coth ka + \frac{1}{2} k\delta \frac{\mu}{\mu_o} \right)^2 \right\}} \quad (10)$$

Note that $k\delta\mu/\mu_o << 1$, this expression is closely approximated by

$$v^c = \frac{\mu k \delta b}{16\eta} \frac{|\hat{K}_o|^2}{\cosh^2 ka} \quad (11)$$

Prob. 9.4.3 (cont.)

This result could have been obtained more simply by approximating $H_x^b \approx 0$ in Eq. 1 and ignoring Eq. 2. That is, the fields in the gap could be approximated as being those for a perfectly conducting fluid.

Prob. 9.4.4 This problem is the same as Problem 9.4.3 except that the uniform magnetic surface force density is given by Eq. 8 from Solution 6.9.2. Thus, shear stress equilibrium for the interface requires that

$$\eta\left[\frac{\partial v_y}{\partial x}\right]^c = \frac{\mu_o}{4}|\hat{H}_o|^2 \frac{\delta}{a} S \tag{1}$$

Using the velocity profile, Eq. 8 from Solution 9.4.3, to evaluate Eq. 1 results in

$$v_c = \frac{\mu_o b}{16\eta}|\hat{H}_o|^2 \frac{\delta}{a} S \tag{2}$$

Prob. 9.5.1 With the skin depth short compared to both the layer thickness and the wavelength, the magnetic fields are related by Eqs. 6.8.5. In the configuration of Table 9.3.1, the origin of the exponential decay is the upper surface, so the solution is translated to $x = \Delta$ and written as

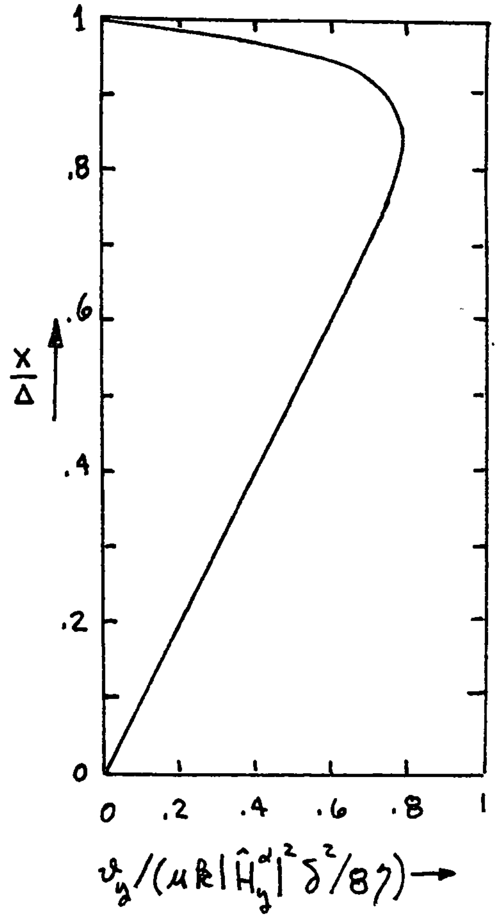
$$\hat{H}_y = \hat{H}_y^d e^{(1+j)(x-\Delta)/\delta} \; ; \; \hat{B}_x^d = \frac{1}{2}(1-j)k\mu\delta\hat{H}_y^d \tag{1}$$

It follows that the time-average magnetic shear stress is

$$T_{yx} = \frac{1}{2}\mathrm{Re}\,\hat{B}_x^d(\hat{H}_y^d)^* = \frac{1}{4}k\mu\delta|\hat{H}_y^d|^2 e^{2(x-\Delta)/\delta} \tag{2}$$

This distribution can now be substituted into Eq. (a) of Table 9.3.1 to obtain the given velocity profile. (b) For $\delta/\Delta = 0.1$, the magnetically induced part of this profile is as sketched in the figure.

Prob. 9.5.1 (cont.)



Prob. 9.5.2 Boundary conditions at the inner and outer wall are

$$\hat{H}_\theta^a = -\hat{K}_o \; ; \; \hat{H}_\theta^b = 0 \tag{1}$$

Thus, from Eq. b of Table 6.5.1, the complex amplitudes of the vector potential are

$$\hat{A}^a = -\mu F_m(b,a,\gamma)\hat{K}_o \; ; \; \hat{A}^b = -\mu G_m(b,a,\gamma)\hat{K}_o \tag{2}$$

In terms of these amplitudes, the distribution of $\hat{A}(r)$ is given by Eq. 6.5.10. In turn, the magnetic field components needed to evaluate the shear stress are now determined.

$$\hat{H}_\theta = -\frac{1}{\mu}\frac{d\hat{A}}{dr} = -\frac{j\gamma}{\mu}\left\{\hat{A}^a\frac{[H_m(j\gamma b)J_m'(j\gamma r) - J_m(j\gamma b)H_m'(j\gamma r)]}{[H_m(j\gamma b)J_m(j\gamma a) - J_m(j\gamma b)H_m'(j\gamma a)]} + \hat{A}^b\frac{[J_m(j\gamma a)H_m'(j\gamma r) - H_m(j\gamma a)J_m'(j\gamma r)]}{[H_m(j\gamma b)J_m(j\gamma a) - H_m(j\gamma a)J_m(j\gamma b)]}\right\} \tag{3}$$

$$\hat{B}_r = -\frac{jm}{r}\hat{A} \tag{4}$$

Thus,

$$T_{\theta r} = \frac{1}{2}\mathrm{Re}\,\hat{B}_r\hat{H}_\theta^* \tag{5}$$

and the velocity profile given by Eq. b of Table 9.3.1 can be evaluated.

Prob. 9.5.2 (cont.)

Because there are rigid walls at r=a and r=b, $v^{\alpha}=0$ and $v^{\beta}=0$.

$$\bar{v} = \bar{i}_{\theta} v \; ; \; v = a \frac{\left(\frac{r}{b} - \frac{b}{r}\right)}{\left(\frac{a}{b} - \frac{b}{a}\right)\eta} \int_b^a \frac{T_{\theta r}}{r} dr - \frac{r}{\eta} \int_b^r \frac{T_{\theta r}}{r} dr \tag{6}$$

The evaluation of these integrals is conveniently carried out numerically, as is the determination of the volume rate of flow Q_v. For a length ℓ in the z direction,

$$Q_v = \ell \int_b^a v \, dr \tag{7}$$

Prob. 9.5.3 With the no slip boundary conditions on the flow, $v^{a}=0$ and $v^{b}=0$, Eq. (c) of Table 9.3.1 gives the velocity profile as

$$v(r) = \frac{1}{4\eta} \frac{\partial p'}{\partial z} \left[(r^2 - b^2) - (a^2 - b^2) \frac{\ln\left(\frac{r}{b}\right)}{\ln\left(\frac{a}{b}\right)} \right] - \frac{1}{\eta} \int_b^r T_{zr} \, dr + \frac{\ln\left(\frac{r}{b}\right)}{\eta \ln\left(\frac{a}{b}\right)} \int_b^a T_{zr} \, dr \tag{1}$$

To evaluate this expression, it is necessary to determine the magnetic stress distribution. To this end, Eq. 6.5.15 gives

$$\hat{\Lambda} = \hat{A}^{a} \frac{\left[H_1(j\gamma b) r J_1(j\gamma r) - J_1(j\gamma b) r H_1(j\gamma r) \right]}{\left[H_1(j\gamma b) J_1(j\gamma a) - J_1(j\gamma b) H_1(j\gamma a) \right]} + \hat{A}^{b} \frac{\left[J_1(j\gamma a) r H_1(j\gamma r) - H_1(j\gamma a) r J_1(j\gamma r) \right]}{\left[J_1(j\gamma a) H_1(j\gamma b) - H_1(j\gamma a) J_1(j\gamma b) \right]} \tag{2}$$

where

$$\hat{B}_r = \frac{j k}{r} \hat{\Lambda}$$

and because $H_z = (1/r)\partial\Lambda/\partial r$, H_z follows as

Prob. 9.5.3 (cont.)

$$H_z = \frac{j\gamma}{\mu}\left\{ \hat{A}^a \frac{[H_1(j\gamma b)J_0(j\gamma r) - J_1(j\gamma b)H_0(j\gamma r)]}{[H_1(j\gamma b)J_1(j\gamma a) - J_1(j\gamma b)H_1(j\gamma a)]} \right. \tag{4}$$

$$\left. + \hat{A}^b \frac{[J_1(j\gamma a)H_0(j\gamma r) - H_1(j\gamma a)J_0(j\gamma r)]}{[J_1(j\gamma a)H_1(j\gamma b) - H_1(j\gamma a)J_1(j\gamma b)]} \right\}$$

Here, Eq. 2.16.26d has been used to simplify the expressions.

Boundary conditions consistent with the excitation and infinitely permeable inner and outer regions are

$$\hat{H}_z^a = \hat{K}_o \quad ; \quad \hat{H}_z^b = 0 \tag{5}$$

Thus, the transfer relations f of Table 6.5.1 give the complex amplitudes needed to evaluate Eqs. (3) and (4).

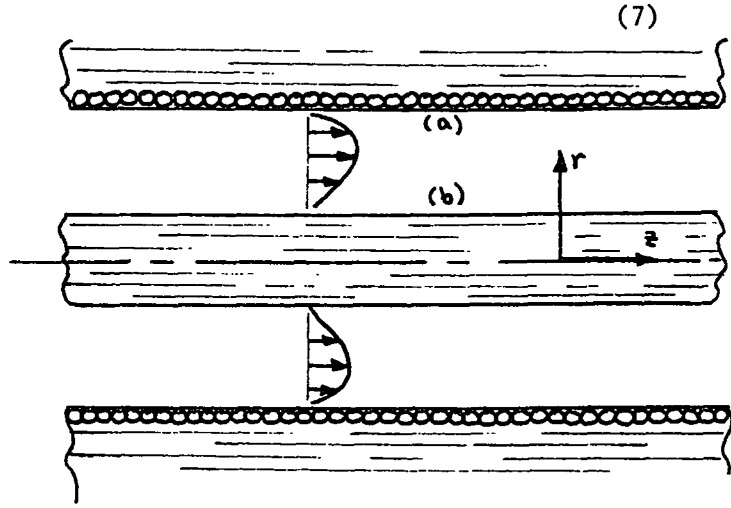
$$\hat{A}^a = \frac{\hat{\Lambda}^a}{a} = -\frac{\mu}{\gamma^2} f_o(b,a,\gamma)\hat{K}_o \; ; \; \hat{A}^b = \frac{\hat{\Lambda}^b}{b} = -\frac{\mu}{\gamma^2} g_o(b,a,\gamma)\hat{K}_o \tag{6}$$

and the required magnetic shear stress follows as

$$T_{zr} = \frac{1}{2}\,\mathrm{Re}\,\hat{B}_r \hat{H}_z \tag{7}$$

The volume rate of flow is related to the axial pressure gradient and magnetic pressure $\mu_o K_o^2$ by integrating Eq. (1).

$$Q_v = \int_b^a v_r \, 2\pi r \, dr \tag{8}$$



<u>Prob. 9.6.1</u> The stress tensor consistent with the force density $F_o\bar{i}_y$ is $T_{yx}=F_o x$. Then, Eq. (a) of Table 9.3.1 with $v^\alpha = 0$ and $v^\beta = 0$, as well as $\partial p'/\partial y = 0$, reflecting the fact that the flow is reentrant, gives the

Prob. 9.6.1 (cont.)

velocity profile

$$v = -\frac{1}{\eta}\int_0^x F_o x' dx' + \frac{x}{\eta\Delta}\int_0^\Delta F_o x' dx' = \frac{F_o\Delta^2}{2\eta}\frac{x}{\Delta}\left(1-\frac{x}{\Delta}\right) \tag{1}$$

For the transient solution, the appropriate plane flow equation is

$$\rho\frac{\partial v}{\partial t} = F_o + \eta\frac{\partial^2 v}{\partial x^2} \tag{2}$$

The particular solution given by Eq. 1 can be subtracted from the total solution with the result that Eq. 2 becomes

$$\rho\frac{\partial v_h}{\partial t} - \eta\frac{\partial^2 v_h}{\partial x^2} = 0 \tag{3}$$

Solutions to this expression of the form $\hat{V}_n(x)\exp s_n t$ must satisfy the equation

$$\frac{d^2\hat{V}_n}{dx^2} + \gamma_n^2\hat{V}_n = 0 \qquad \text{where} \quad \gamma_n^2 \equiv -\frac{\rho s_n}{\eta} \tag{4}$$

The particular solution already satisfies the boundary conditions. So must the homogeneous solution. Thus, to satisfy boundary conditions $v(0,t)=0,\ v(\Delta,t)=0$

$$v_h = \sum_{n=1}^{\infty} \mathrm{Re}\,\hat{V}_n \sin(\gamma_n x)\,e^{s_n t} \quad ; \quad \gamma_n = \frac{n\pi}{\Delta} \tag{5}$$

To satisfy the initial conditions, $v_y(x,0) = v_{part} + v_h(x,0) = 0$ and so

$$\sum_{n=1}^{\infty} \mathrm{Re}\,\hat{V}_n \sin(\gamma_n x)\,e^{s_n t} = -\frac{F_o\Delta^2}{2\eta}\frac{x}{\Delta}\left(1-\frac{x}{\Delta}\right) \tag{6}$$

Multiplication by sin $(m\pi x/\Delta)$ and integration from x=0 to x= Δ serves to evaluate the Fourier coefficients. Thus, the transient solution is

$$v(x,t) = \frac{F_o\Delta^2}{2\eta}\left\{\left(\frac{x}{\Delta}\right)\left(1-\frac{x}{\Delta}\right) - 4\sum_{n=1}^{\infty}\left(\frac{1-\cos n\pi}{(n\pi)^3}\right)\sin\left(\frac{n\pi x}{\Delta}\right)e^{-\left[\frac{\eta}{\rho}\left(\frac{n\pi}{\Delta}\right)^2\right]t}\right\} \tag{7}$$

Although it is the viscous diffusion time that determines how long is required for the fully developed flow to be established, the viscosity is

Prob. 9.6.1 (cont.)

not involved in determining how quickly the bulk of the fluid will respond. Because the force is distributed throughout the bulk, it is the fluid inertia that determines the degree to which the fluid will in general respond. This can be seen by taking the limit of Eq. 7 where times are short compared to the viscous diffusion time and the exponential can be approximated by the first two terms in the series expansion. Then, for $(\eta/\rho)(\pi/\Delta)^2 t << 1$,

$$v(x,t) \rightarrow \left[\frac{2F_o}{\rho}\sum_{n=1}^{\infty}\left(\frac{1-\cos n\pi}{n\pi}\right)\sin\frac{n\pi x}{\Delta}\right]t \tag{8}$$

which is what would be expected by simply equating the mass times acceleration of the fluid to the applied force.

<u>Prob. 9.6.2</u> The general procedure for finding the temporal transient outlined with Prob. 9.6.2 makes clear what is required here. If the profile is to remain invariant, then the fully developed flow must have the same profile as the transient or homogeneous part at any instant. The homogeneous response takes the form of Eq. 5 from the solution to Prob. 9.6.1. For the fully developed flow to have the same profile requires

$$v_{fd} = \frac{F_n}{\eta}\left(\frac{\Delta}{n\pi}\right)^2 \sin\left(\frac{n\pi}{\Delta}x\right) \tag{1}$$

where the coefficient has been adjusted so that the steady force equation is satisfied with the force density given by

$$F_o = F_n \sin\left(\frac{n\pi}{\Delta}x\right) \tag{2}$$

The velocity temporal transient is then the sum of the fully developed and the homogeneous solutions, with the coefficient in front of the latter adjusted to make $v(x,0)=0$.

Prob. 9.6.2 (cont.)

$$v = \frac{F_n}{\eta}\left(\frac{\Delta}{n\pi}\right)^2 \sin\frac{n\pi}{\Delta}x\left(1 - e^{s_n t}\right);\ s_n \equiv -\frac{\eta}{\rho}\left(\frac{n\pi}{\Delta}\right)^2 \qquad (3)$$

Thus, if the force distribution is the same as any one of the eigenmodes, the resulting velocity profile will remain invariant.

Prob. 9.7.1 The boundary layer equations again take the similarity form of Eqs. 17. However, the boundary conditions are

$$v_x(0,y)=0 \Rightarrow f(0)=0;\ v_y(0,y)=U \Rightarrow g(0)=-2;\ v_y(\infty,y)=0 \Rightarrow g(\infty)\to 0 \qquad (1)$$

where U now denotes the velocity in the y direction adjacent to the plate. The resulting distributions of f, g and h are shown in Fig. P9.7.1. The condition as $\xi\to\infty$ is obtained by iterating with h(0) to obtain h(0) =

Thus, the viscous shear stress at the boundary is (Eq. 19)

$$S_{yx}(0,y) = \frac{1}{4}U\eta\sqrt{\frac{\rho U}{\eta y}}\ h(0) = \qquad (2)$$

and it follows that the total force on a length L of the plate is

$$f_y = w\int_0^L S_{yx}(0,y)\,dy = \frac{h(0)}{2}wU\sqrt{\eta\rho UL} \qquad (3)$$

Prob. 9.7.2 What is expected is that the similarity parameter, ξ, is essentially

$$\sqrt{\frac{\tau_v}{\tau_t}} = \sqrt{\frac{\rho x^2}{\eta \tau_t}} \tag{1}$$

where τ_t is the time required for a fluid element at the interface to reach the position y. Because the interfacial velocity is not uniform, this time must be found. In Eulerian coordinates, the interfacial velocity is given by Eq. 9.7.28.

$$v_y = K y^{1/3}; \quad K \equiv \left(\frac{T_o^2}{\rho \eta}\right)^{1/3} 1.296 \tag{2}$$

(2)

For a particle having the position y, it follows that

$$\frac{dy}{dt} = K y^{1/3} \Rightarrow \frac{dy}{y^{1/3}} = K\,dt \tag{3}$$

and integration gives

$$\int_0^y y^{-1/3} dy = K \int_0^{\tau_t} dt \Rightarrow \tau_t = \frac{3}{2} y^{2/3} / K \tag{4}$$

Substitution into Eq. 1 then gives

$$\sqrt{\frac{\tau_v}{\tau_t}} = \sqrt{\frac{2(1.296)}{3}} \left(\frac{T_o \rho}{\eta^2}\right)^{1/3} \times y^{-\frac{1}{3}} \tag{5}$$

In the definition of the similarity parameter, Eq. 25, the numerical factor has been set equal to unity.

Prob. 9.7.3 Similarity parameter and function are assumed to take the forms given by Eq. 23. The stress equilibrium at the interface, $S_{yx}(x=0) = -T(y)$, requires that

$$-\frac{T_o}{a^k} y^k = -\eta c_1^2 c_2 y^{m+2n} f'' \tag{1}$$

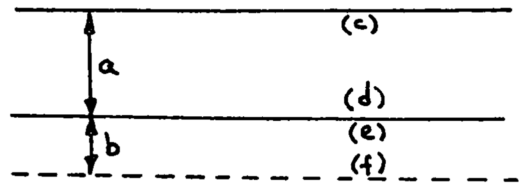
so that m+2n=k and $\eta c_1^2 c_2 = -T_o/a^k$. Substitution into Eq. 14 shows that for the similarity solution to be valid, 2m+2n-1 = m+3n or m=n+1. Thus, it follows that $n=(k-1)/3$ and $m=(k+2)/3$. If $(\eta/\rho)(c_1/c_2) = -1$, the boundary layer equation then reduces to

$$f''' - \frac{(2k+1)}{3}(f')^2 + m f f'' = 0 \tag{2}$$

which is equivalent to the given system of first order equations. The only boundary condition that appears to be different from those of Eq. 27 is on the interfacial shear stress. However, with the parameters as defined, Eq. 1 reduces to simply $h(0)=-1$.

Prob. 9.7.4 (a) In the liquid volume, the potential must satisfy Laplace's equation, which it does. It also satisfies the boundary condition imposed on the potential by the lower electrodes. At the upper interface, the electric field is $\bar{E} = V_b y/b^2$, which satisfies the condition that there be no normal electric field (and hence current density) at the interface. (b) With the given potential at x=a, the x directed electric field is the potential difference divided by the spacing: $E_x = [-V_b y^2/2b^2 + V_a y^2 2b^2]/a$. Thus, the surface force density is $T = \epsilon_o E_x E_y = (\epsilon_o V_a/2b^4 a)(V_a - V_b) y^3$. (c) With the identification $T_o/a^k \rightarrow (\epsilon_o V_a/2b^4 a)(V_a - V_b)$ and k=3, the surface force density takes the form assumed in Prob. 9.7.3.

Prob. 9.8.1 First, determine the electric fields and hence the surface force density. The applied potential can be written in the complex notation as $\Phi^f = \mathrm{Re}\, \frac{V_o}{2}\left(e^{-j\beta y} + e^{j\beta y}\right)$ so that the desired standing wave solution is the superposition of two traveling wave solutions with amplitudes $\tilde{\Phi}^f_{\pm} = V_o/2$. Boundary conditions are



$$\tilde{J}_x^c = 0\,,\ \tilde{\Phi}^d = \tilde{\Phi}^e\,,\ \sigma_a \tilde{E}_x^d = \sigma_b \tilde{E}_x^e\,,\ \tilde{\Phi}^f = \frac{V_o}{2} \tag{1}$$

And bulk transfer relations are (Eqs. (a), Table 2.16.1)

$$\begin{bmatrix} \tilde{E}_x^c \\ \tilde{E}_x^d \end{bmatrix} = \beta \begin{bmatrix} -\coth\beta a & \dfrac{1}{\sinh\beta a} \\ \dfrac{-1}{\sinh\beta a} & \coth\beta a \end{bmatrix} \begin{bmatrix} \tilde{\Phi}^c \\ \tilde{\Phi}^d \end{bmatrix} \tag{2}$$

$$\begin{bmatrix} \tilde{E}_x^e \\ \tilde{E}_x^f \end{bmatrix} = \beta \begin{bmatrix} -\coth\beta b & \dfrac{1}{\sinh\beta b} \\ \dfrac{-1}{\sinh\beta b} & \coth\beta b \end{bmatrix} \begin{bmatrix} \tilde{\Phi}^e \\ \tilde{\Phi}^f \end{bmatrix} \tag{3}$$

It follows that

$$\tilde{\Phi}^e = \frac{V_o\,\sigma_b}{2\sinh\beta b\,(\sigma_a \tanh\beta a + \sigma_b \coth\beta b)} \tag{4}$$

where then

$$\tilde{E}_z^e = j\beta\tilde{\Phi}^e\ ;\ \epsilon_a \tilde{E}_x^d = \frac{\epsilon_a\beta\tilde{\Phi}^e}{\coth\beta a}\ ;\ \epsilon_b \tilde{E}_x^e = \frac{\epsilon_b\beta\tilde{\Phi}^e}{\coth\beta a}\,\frac{\sigma_a}{\sigma_b} \tag{5}$$

Now, observe that $\tilde{E}_x$ and $\tilde{\Phi}^e$ are real and even in β while E_z is imaginary and odd in β. Thus, the surface force density reduces to

$$T_y = -\left(\epsilon_a \tilde{E}^d_{x+} - \epsilon_b \tilde{E}^d_{x+}\right) j\, \tilde{E}^e_{z+}\ 2\sin 2\beta y \tag{6}$$

and evaluation gives $T_y = T_o \sin 2\beta y$

$$T_o \equiv \frac{\beta^2 V_o^2\, \sigma_b\,(\epsilon_a\sigma_b - \epsilon_b\sigma_a)}{2\sinh^2\beta b\,(\sigma_a\tanh\beta a + \sigma_b\coth\beta b)^2 \coth\beta a} \tag{7}$$

Prob. 9.8.1 (cont.)

The mechanical boundary conditions consistent with the assumption that gravity holds the interface flat are

$$\tilde{v}_x^c = 0, \tilde{v}_y^c = 0, \tilde{v}_x^d = 0, \tilde{v}_x^e = 0, \tilde{v}_y^d = \tilde{v}_y^e, \tilde{v}_x^f = 0, \tilde{v}_y^f = 0 \tag{8}$$

Stress equilibrium for the interface requires that

$$T_y + S_{yx}^d - S_{yx}^e = 0 \tag{9}$$

In terms of the complex amplitudes, this requires

$$\pm j\frac{T_o}{2} + \tilde{S}_{yx\pm}^d - S_{yx\pm}^e = 0 \tag{10}$$

With the use of the transfer relations from Sec. 7.20 for cellular creep flow, Eqs. 7.20.6, this expression becomes

$$\pm j\frac{T_o}{2} + (\eta_a P_{44}^a - \eta_b P_{33}^b)\tilde{v}_{y\pm}^d = 0 \tag{11}$$

and it follows that the velocity complex amplitudes are

$$\tilde{v}_{y\pm}^d = \mp j\frac{T_o}{2(\eta_a P_{44}^a - \eta_b P_{33}^b)} \tag{12}$$

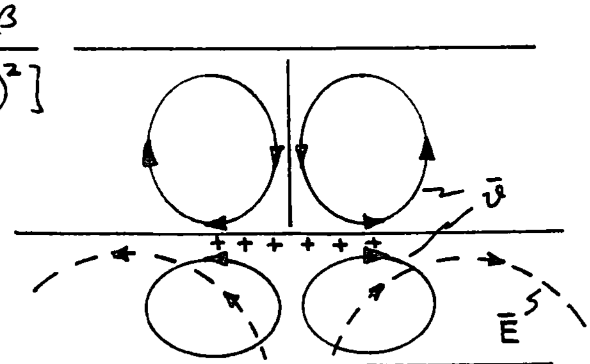
The actual interfacial velocity can now be stated

$$v_y = Re(\tilde{v}_{y+}e^{-j\beta y} + \tilde{v}_{y-}e^{j\beta y}) = \frac{T_o \sin 2\beta y}{-\eta_a P_{44}^a + \eta_b P_{33}^b} \tag{13}$$

where, from Eq. 7.20.6,

$$P_{44}^a = -\frac{\left[\frac{1}{4}\sinh 4\beta a - \beta a\right]8\beta}{\left[\sinh^2 2\beta a - (2\beta a)^2\right]}$$

$$P_{33}^b = \frac{\left[\frac{1}{4}\sinh 4\beta b - \beta b\right]8\beta}{\left[\sinh^2 2\beta b - (2\beta b)^2\right]}$$



Note that P_{44} and P_{33} are positive. Thus, the coefficient of sin $2\beta y$ is positive and circulations are as sketched and as would be expected in view of the sign of σ_f and E_z at the interface.

Prob. 9.8.1 (cont.)

Charge conservation, including the effect of charge convection at the interface, is represented by the boundary condition

$$\sigma_a E_x^d - \sigma_b E_x^e + \frac{\partial}{\partial y}\left[\left(\epsilon_a E_x^d - \epsilon_b E_x^e\right) v_y\right] = 0 \tag{14}$$

The convection term will be negligible if

$$\frac{\epsilon}{\sigma} v_y \beta << 1 \tag{15}$$

where ϵ/σ is the longest time constant formed from ϵ_a, ϵ_b and σ_a, σ_b. (A more careful comparison of terms would give a more specific combination of ϵ's and σ's in forming this time constant.) The velocity is itself a function of three lengths, $2\pi/\beta$, a and b. With the assumption that βa and βb are of the order of unity, the velocity given by Eqs. 13 and 7 is typically $\epsilon(\beta V_o)^2/\eta\beta$ and it follows that Eq. 15 takes the form of a condition on the ratio of the charge relaxation time to the electroviscous time.

$$\frac{\epsilon}{\sigma} \Big/ \frac{\eta}{(\beta V_o)^2 \epsilon} << 1 \tag{16}$$

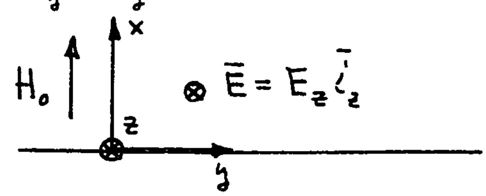
Effects of inertia are negligible if the inertial and viscous force densities bare the relationship

$$|\rho \bar{v}\cdot\nabla\bar{v}| << \eta|\nabla^2 \bar{v}| \Rightarrow \frac{\rho v_y}{\beta\eta} << 1 \tag{17}$$

With the velocity again taken as being of the order of $\epsilon(\beta V_o)^2/\eta\beta$, this condition results in the requirement that the ratio of the viscous diffusion time to the electroviscous time be small.

$$\frac{\rho}{\eta\beta^2} \Big/ \frac{\eta}{(\beta V_o)^2 \epsilon} << 1 \tag{18}$$

Prob. 9.9.1 The flow is fully developed, so $\bar{v} = v_y(x)\bar{i}_y$



Thus, inertial terms in the Navier-Stoke's equation are absent. The x and y components of that equation therefore become

$$\frac{\partial p}{\partial x} = 0 \tag{1}$$

$$\frac{\partial p}{\partial y} = \eta\frac{\partial^2 v_y}{\partial x^2} + \sigma\,(E_z - v_y\mu_o H_o)\mu_o H_o \tag{2}$$

Because E_z is independent of x, this expression is written in the form

$$\frac{d^2 v_y}{dx^2} - \frac{\sigma}{\eta}(\mu_o H_o)^2 v_y = \left(\frac{\partial p}{\partial y} - \sigma\mu_o H_o\right)\frac{1}{\eta} \tag{3}$$

so that what is on the right is independent of x. Solutions to this expression that are appropriate for the infinite half space are exponentials. The growing exponential is excluded, so the homogeneous solution is $\exp -\gamma x$ where $\gamma \equiv \mu_o H_o\sqrt{\frac{\sigma}{\eta}}$

The particular solution is $\left(-\frac{\partial p}{\partial y} + \sigma\mu_o H_o E_z\right)/\sigma(\mu_o H_o)^2$. The combination of these that makes $v_y=0$ at the wall where x=0 is

$$v_y = \left(\sigma\mu_o H_o E_z - \frac{\partial p}{\partial y}\right)\frac{\left(1 - e^{-\gamma x}\right)}{\sigma(\mu_o H_o)^2} \tag{4}$$

Thus, the boundary layer has a thickness that is approximately γ^{-1}.

Prob. 9.14.1 There is no electromechanical coupling, so $\mathcal{E}=0$ and Eq. 3 becomes $p = -\rho g(x-\xi)$. Thus, Eq. 5 becomes $p + \rho g x = \rho g\xi$ and in turn Eq. 4 is

$$\rho\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial y}\right) + \rho g\frac{\partial \xi}{\partial y} = 0 \tag{1}$$

Because $A = \xi - \Xi$, Eq. 9 is

$$\frac{\partial \xi}{\partial t} + \frac{\partial}{\partial y}\left[(\xi - \Xi)v\right] = 0 \tag{2}$$

In the steady state, Eq. 2 shows that

$$(\xi - \Xi)v = \xi_\infty v_\infty \tag{3}$$

while Eq. 1 gives

$$\frac{d}{dy}\left(\frac{1}{2}\rho v^2 + \rho g\xi\right) = 0 \Rightarrow \frac{1}{2}\rho v^2 + \rho g\xi = \frac{1}{2}\rho v_\infty^2 + \rho g\xi_\infty \tag{4}$$

Combined, these expressions show that

$$H \equiv \frac{1}{2}\rho\frac{\xi_\infty^2 v_\infty^2}{(\xi - \Xi)^2} + \rho g\xi = \frac{1}{2}\rho v_\infty^2 + \rho g\xi_\infty \tag{5}$$

The plot of this function with the bottom elevation Ξ (y) as a parameter is

Prob. 9.14.1 (cont.)

shown in the figure. The Flow conditions establish the vertical line along which the transition must evolve. Given the bottom elevation and hence the particular curve,

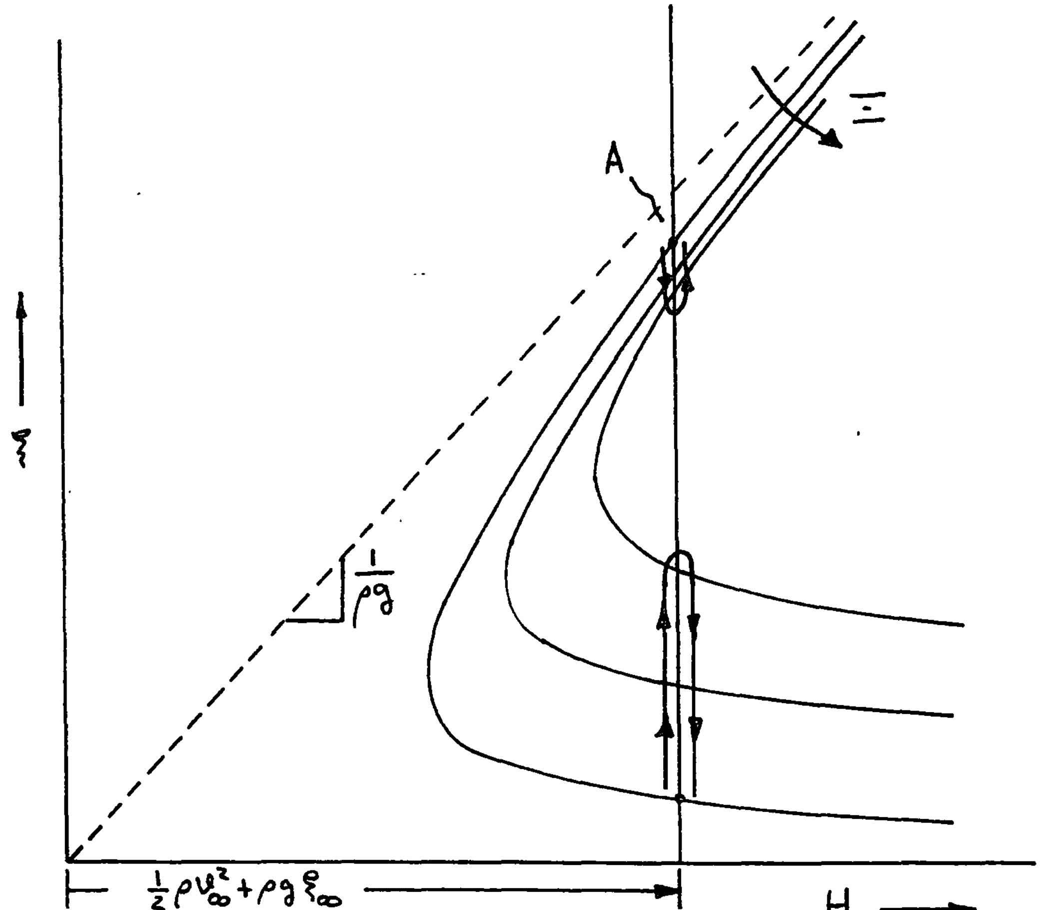


Fig. S9.14.1

the local depth follows from the intersection with the vertical line. If the flow is initiated above the minimum in H(ξ), the flow enters subcritical, whereas if it enters below the minimum ($\xi < \xi_c$), the flow enters supercritical. This can be seen by evaluating

$$\frac{dH}{d\xi} = 0 \Rightarrow (\xi_c - \Xi)^3 = \frac{\xi_\infty^2 v_\infty^2}{g} \tag{8}$$

and observing that the critical depth in the figure comes at

$$\frac{(\xi_c - \Xi)}{\xi_\infty} = \frac{v_\infty}{\sqrt{g(\xi_c - \Xi)}} \tag{9}$$

Consider three types of conservative transitions caused by having a positive bump in the bottom. For a flow initiated at A, the depth decreases where Ξ increases and then returns to its entrance value, as shown in Fig. 2a. For flow entering with depth at B, the reverse is true. The depth increases where the bump occurs. These situations are distinguished

Prob. 9.14.1 (cont.)

by what the entrance depth is relative to the critical depth, given by Eq. (9). If the entrance depth ξ_a is greater than critical, $(\xi_c - \Xi)$, then it follows from Eq. 9 that the entrance velocity, v_ω, is less than the gravity wave velocity $\sqrt{g(\xi_c - \Xi)}$ for the critical depth. A third possibility is that a flow initiated at A reaches the point of tangency between the vertical line and the head curve. Eqs. 3 and 8 combine to show that

$$v = \sqrt{g(\xi - \Xi)} \qquad (10)$$

Then, critical conditions prevail at the peak of the bump and the flow can continue into the subcritical regime, as sketched in Fig. 2c. A similar super-subcritical transition is also possible. (See Rouse, H., Elementary Mechanics of Fluids, John Wiley & Sons, N.Y. (1946), p. 139.

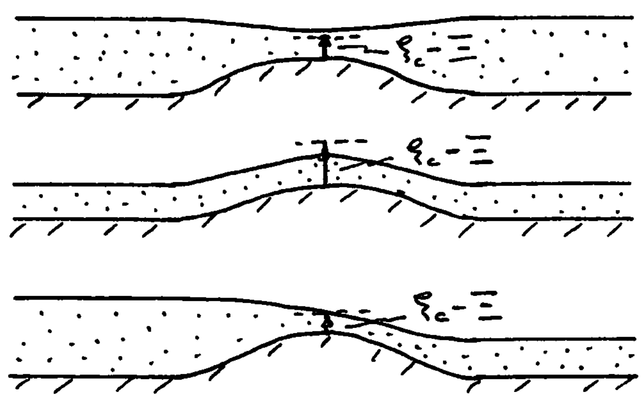


Fig. S9.14.2

Prob. 9.14.2 The normalized mass conservation and momentum equations are

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \qquad (1)$$

$$\left(\frac{d}{\ell}\right)^2 \left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y}\right) + \frac{\partial p}{\partial x} = -1 \qquad (2)$$

$$\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + \frac{\partial p}{\partial y} = 0 \qquad (3)$$

Thus, to zero order in $(d/\ell)^2$, the vertical force equation reduces to a static equilibrium; $p = -\rho g(x - \xi)$. The remaining two expressions then comprise the fundamental equations. Observe that these expressions in themselves do not require that $v_y = v_y(y,t)$. In fact, the quasi-one-dimensional model allows rotational flows. However, if it is specified that the flow is irrotational to begin with, then it follows from Kelvin's Theorem on vorticity that the flow remains irrotational. This is a result of the expressions above, but is best seen in general. The condition of irrota-

Prob. 9.14.2 (cont.)

tionality in dimensionless form is

$$\left(\frac{d}{\ell}\right)^2 \frac{\partial v_x}{\partial y} = \frac{\partial v_y}{\partial x} \tag{4}$$

and hence the quasi-one-dimensional space-rate expansion, to zero order, requires that $v_y=v_y(y,t)$. Thus, Eqs. 1 and 2 become the fundamental laws for the quasi-one-dimensional model

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \tag{5}$$

$$\frac{\partial v_y}{\partial t} + v_y \frac{\partial v_y}{\partial y} + \frac{\partial p}{\partial y} = 0 \tag{6}$$

with the requirements that p is determined by the transverse static equilibrium and $v_y=v_y(y,t)$.

Prob. 9.14.3 With gravity ignored, the pressure is uniform over the liquid cross-section. This means that it is the same pressure that appears in the normal stress balance for each of the interfaces.

$$\frac{1}{2}(\epsilon-\epsilon_o)\left(\frac{V_a}{\beta\xi_b}\right)^2 = -p = \frac{1}{2}(\epsilon-\epsilon_o)\left(\frac{V_a}{\alpha\xi_a}\right)^2 \tag{1}$$

It follows that the interfacial positions are related.

$$\beta\xi_b = \alpha\xi_a \tag{2}$$

Within a constant associated with the fluid in the neighborhood of the origin, the cross-sectional area is then

$$A = \pi\xi_a^2\left(\frac{\alpha}{2\pi}\right) + \pi\xi_b^2\left(\frac{\beta}{2\pi}\right) = \frac{\alpha}{2}\left(1+\frac{\alpha}{\beta}\right)\xi_a^2 \tag{3}$$

or essentially represented by the variable ξ_a^2. Mass conservation, Eq. 9.13.9, gives

$$\frac{\partial \xi_a^2}{\partial t} + \frac{\partial}{\partial z}\left(v\xi_a^2\right) = 0 \tag{4}$$

Because the pressure is uniform throughout, Eq. 9.13.3 is simply the force balance equation for the interface (either one).

$$p = -\frac{1}{2}(\epsilon-\epsilon_o)\frac{V_a^2}{\alpha^2\xi_a^2} \tag{5}$$

Thus, the force equation, Eq. 9.13.4, becomes the second equation of motion.

$$\rho\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial z}\right) + \frac{1}{2}(\epsilon-\epsilon_o)\frac{V_a^2}{\alpha^2}\frac{1}{(\xi_a^2)^2}\frac{\partial^2\xi_a^2}{\partial z^2} = 0 \tag{6}$$

Prob. 9.16.1   Substitution of Eq. 8 for T in Eq. 2 gives

$$c_p T_o \left(\frac{v A}{v_o A_o}\right)^{1-\gamma} + \frac{1}{2} v^2 = c_p T_o + \frac{1}{2} v_o^2 \tag{1}$$

Manipulation then results in

$$\left(\frac{v}{v_o}\right)\left(\frac{A}{A_o}\right) = \left\{ 1 + \frac{v_o^2}{2 c_p T_o}\left[1 - \left(\frac{v}{v_o}\right)^2\right]\right\}^{\frac{1}{1-\gamma}} \tag{2}$$

Note that

$$\frac{v_o^2}{2 c_p T_o} = \frac{\gamma R v_o^2}{2 c_p \gamma R T_o} = \frac{\gamma R}{2 c_p} M_o^2 = \frac{\frac{c_p}{c_v} R}{2 c_p} M_o^2 = (\gamma - 1)\frac{M_o^2}{2} \tag{3}$$

where use has been made of the relations $\gamma \equiv c_p/c_v$ and $R = c_p - c_v$ and it follows that Eq. 2 is

$$\left(\frac{v}{v_o}\right)\left(\frac{A}{A_o}\right) = \left\{ 1 + (\gamma - 1)\frac{M_o^2}{2}\left[1 - \left(\frac{v}{v_o}\right)^2\right]\right\}^{\frac{1}{1-\gamma}} \tag{4}$$

so that the required relation, Eq. 9, results.

Prob. 9.16.2 The derivative of Eq. 9.16.9 that is required to be zero is

$$\frac{d(A/A_o)}{d(v/v_o)} = -\left(\frac{v_o}{v}\right)^2\left\{ 1 + (\gamma-1)\frac{M_o^2}{2}\left[1 - \left(\frac{v}{v_o}\right)^2\right]\right\}^{1/(1-\gamma)} + M_o^2\left\{ 1 + (\gamma-1)\frac{M_o^2}{2}\left[1 - \left(\frac{v}{v_o}\right)^2\right]\right\}^{\frac{1}{1-\gamma} - 1} \tag{1}$$

This expression can be factored and written as

$$\frac{d(A/A_o)}{d(v/v_o)} = \left(\frac{v_o}{v}\right)^2\left\{ 1 + (\gamma-1)\frac{M_o^2}{2}\left[1 - \left(\frac{v}{v_o}\right)^2\right]\right\}^{\frac{1}{1-\gamma}} \cdot \left\{ -1 + \left(\frac{v}{v_o}\right)^2 M_o^2\left[1 + (\gamma-1)\frac{M_o^2}{2}\left[1 - \left(\frac{v}{v_o}\right)^2\right]\right]^{-1}\right\} \tag{2}$$

By definition, the Mach number is

$$M \equiv \frac{v}{\sqrt{\gamma R T}} \quad ; \quad M_o \equiv \frac{v_o}{\sqrt{\gamma R T_o}} \tag{3}$$

Thus,

$$\frac{M}{M_o} = \frac{v}{v_o}\sqrt{\frac{T_o}{T}} \tag{4}$$

Through the use of Eq. 9.16.8, this expression becomes

$$\frac{M}{M_o} = \left(\frac{v}{v_o}\right)\left\{ 1 + (\gamma-1)\frac{M_o^2}{2}\left[1 - \left(\frac{v}{v_o}\right)^2\right]\right\}^{-\frac{1}{2}} \tag{5}$$

Substitution of the quantity on the left for the group on the right as it

Prob. 9.16.2 (cont.)

appears in Eq. 2 reduces the latter expression to

$$\frac{d(A/A_o)}{d(v/v_o)} = \left(\frac{v_o}{v}\right)^2 \left\{ 1 + (\gamma-1)\frac{M_o^2}{2}\left[1 - \left(\frac{v}{v_o}\right)^2\right]\right\}^{\frac{1}{1-\gamma}} (-1 + M^2) \tag{6}$$

Thus, the derivative is zero at M = 1.

<u>Prob. 9.16.3</u> Eqs. (c) and (e) require that

$$\frac{d}{dz}(c_p T) = -\frac{d}{dz}\left(\frac{1}{2}v^2\right) \tag{1}$$

so that the force equation becomes

$$\frac{dp}{dz} = -\rho v \frac{dv}{dz} = -\rho \frac{d}{dz}\left(\frac{1}{2}v^2\right) = \rho c_p \frac{dT}{dz} \tag{2}$$

In view of the mechanical equation of state, Eq. (d), this relation becomes

$$\frac{dp}{dz} = \rho \frac{d}{dz}\left(\frac{c_p p}{\rho R}\right) = \frac{1}{1-\gamma}\rho \frac{d}{dz}\left(\frac{p}{\rho}\right) = \frac{1}{1-\gamma}\left(\frac{dp}{dz} - \frac{p}{\rho}\frac{d\rho}{dz}\right) \tag{3}$$

With the respective derivatives placed on opposite sides of the equation, this expression becomes

$$\gamma \frac{dp}{p} = \frac{d\rho}{\rho} \tag{4}$$

and hence integration results in the desired isentropic equation of state.

$$\frac{p}{p_o} = \left(\frac{\rho}{\rho_o}\right)^{-\gamma} \tag{5}$$

Prob. 9.17.1 Equations (a)-(e) of Table 9.15.1 with F and EJ provided by Eqs.5, 7 and 8 are the starting relations

$$\rho v A = \rho_o v_o A_o \tag{1}$$

$$\rho v \frac{dv}{dz} + \frac{dp}{dz} = -\sigma v B^2(1-K) \tag{2}$$

$$\rho \frac{d}{dz}\left(c_p T + \frac{1}{2}v^2\right) = -\sigma v B^2(1-K)K \tag{3}$$

$$p = \rho R T \tag{4}$$

That the Mach number remains constant requires that

$$v^2/\gamma R T = M_o^2 \tag{4}$$

and differentiation of this relation shows that

$$2 v\, dv = \gamma R M_o^2\, dT \tag{5}$$

Substitute for ρ in Eqs. 2 and 3 using Eq. 4. Then multiply Eq. 2 by -K and add to Eq. 3 to obtain

$$\frac{p}{RT}(1-K)\frac{d}{dz}\left(\frac{1}{2}v^2\right) - K\frac{dp}{dz} + \frac{c_p p}{RT}\frac{dT}{dz} = 0 \tag{6}$$

In view of the constraint from Eq. 5, the first term can be expressed as a function of T

$$\frac{p}{RT}\left[c_p + (1-K)\frac{\gamma R M_o^2}{2}\right]\frac{dT}{dz} - K\frac{dp}{dz} = 0 \tag{7}$$

Then, division by p and rearrangement gives

$$\alpha \frac{dT}{T} = \frac{dp}{p} \tag{8}$$

where

$$\alpha \equiv \frac{\gamma}{K(\gamma-1)}\left[1 - \frac{1}{2}(\gamma-1)M_o^2(K-1)\right]$$

Hence,

$$\frac{p}{p_o} = \left(\frac{T}{T_o}\right)^{\alpha} \tag{9}$$

In turn, it follows from Eq. 4 that

$$\frac{\rho}{\rho_o} = \left(\frac{T}{T_o}\right)^{\alpha-1} \tag{10}$$

Prob. 9.17.1 (cont.)

The velocity is already determined as a function of T by Eq. 4.

$$\frac{v}{v_o} = \sqrt{\frac{T}{T_o}} \tag{11}$$

Finally, the area follows from Eq. 1 and these last two relations.

$$\frac{A}{A_o} = \left(\frac{T_o}{T}\right)^{\alpha - \frac{1}{2}} \tag{12}$$

The key to now finding all of the variables is T(z), which is now found by substituting Eq. 11 into the energy equation, Eq. 3

$$\left(c_p T_o + \frac{v_o^2}{2}\right)\left(\frac{T}{T_o}\right)^{\alpha - 3/2} \frac{d}{dz}\left(\frac{T}{T_o}\right) = -\sigma \frac{v_o}{\rho_o} B^2 (1 - K) K \tag{13}$$

This expression can be integrated to provide the temperature evolution with z.

$$\frac{T}{T_o} = \left\{ 1 - \left[\frac{\sigma B^2 (1-K) K (\alpha - \frac{1}{2}) v_o}{(c_p T_o + \frac{1}{2} v_o^2)\rho_o}\right] z \right\}^{1/(\alpha - \frac{1}{2})} \tag{14}$$

Given this expression for T(z), the other variables follow from Eqs. 9-12. The specific entropy is also now evaluated. Equation 7.23.12 is evaluated using Eqs. 9 and 10 to obtain

$$s_T = s_T^o + c_v[\alpha - \gamma(\alpha - 1)] \ln\left[\frac{T}{T_o}\right] \tag{15}$$

Note that $c_v[\alpha - \gamma(\alpha - 1)] = c_p - \alpha R$.

Prob. 9.17.2 First arrange the conservation equations as given.

Conservation of mass, Eq. (a) of Table 9.15.1, is

$$\frac{d}{dz}(\rho v A) = A\left(\rho \frac{dv}{dz} + v\frac{d\rho}{dz}\right) + \rho v \frac{dA}{dz} = 0 \tag{1}$$

Conservation of momentum is Eq. (b) of that table with F given by Eq. 9.17.4.

$$\rho v \frac{dv}{dz} + \frac{dp}{dz} = \sigma B(E + vB) \tag{2}$$

Conservation of energy is Eq. (c), JE expressed using Eq. 9.17.5.

$$\rho v \frac{d}{dz}\left(c_p T + \frac{1}{2}v^2\right) = -\sigma E\,(E + vB) \tag{3}$$

Because $\gamma = c_p/c_v$, $R = c_p - c_v$, and $M^2 = v^2/\gamma R T$, this expression becomes

$$\rho v^3 \frac{1}{v}\frac{dv}{dz} + \rho v c_p \frac{dT}{dz} = \rho v^3 \frac{1}{v}\frac{dv}{dz} + \frac{\rho v^3}{M^2(\gamma-1)}\frac{1}{T}\frac{dT}{dz} = \sigma E(E+vB) \tag{4}$$

The mechanical equation of state becomes

$$p = \rho R T \rightarrow \frac{1}{p}\frac{dp}{dz} + \frac{1}{\rho}\frac{d\rho}{dz} + \frac{1}{T}\frac{dT}{dz} = 0 \tag{5}$$

Finally, from the definition of M^2,

$$\frac{d}{dz}M^2 = \frac{d}{dz}\left(\frac{v^2}{\gamma R T}\right) \Rightarrow \frac{M^{2\prime}}{M^2} + \frac{T'}{T} - \frac{2v'}{v} = 0 \tag{6}$$

Arranged in matrix form, Eqs. 1,2,4,5 and 6 are the expression summarized in the problem statement.

The matrix is inverted by using Cramer's rule. As a check in carrying out this inversion, the determinant of the matrix is

$$Det = (1 - M^2)\frac{M^2\gamma^2 p^4}{\gamma - 1} \tag{7}$$

Integration of this system of first order equations is straightforward if conditions at the inlet are given. (Numerical integration can be carried out using standard packages such as the Fortran IV IMSL Integration Package DEVREK.) As suggested by the discussion in Sec. 9.16, whether the flow is "super-critical" or "sub-critical" will play a role in determining cause and effect and hence in establishing the appropriate boundary conditions. When the channel is fitted into a system, it is in general necessary to meet conditions at the downstream end. This could be done by using one or more of the upstream conditions as interation variables. This technique is familiar from the integration of boundary layer equations in Sec. 9.7.

Prob. 9.17.2 (cont.)

If the channel is to be designed to have a given distribution of one of the variables on the left, with the channel area to be so determined, these expressions should be rewritten with that variable on the right and A'/A on the left. For example, if the mach number is a given function of z, then the last expression can be solved for A'/A as a function of $(M^2)'/M^2$, $\sigma B(E+vB)$ and $\sigma E(E+vB)$. The other expressions can be written in terms of these same variables by substituting for A'/A with this expression.

Prob. 9.17.3 From Prob. 9.17.2, $A'=0$, reduces the transition equations to $[J=\sigma(E+vB)]$.

$$\begin{bmatrix} \frac{\rho'}{\rho} \\ \frac{p'}{p} \\ \frac{v'}{v} \\ \frac{T'}{T} \\ \frac{(M^2)'}{M^2} \end{bmatrix} = \frac{1}{1-M^2}\begin{bmatrix} -\frac{1}{p} & -\frac{(\gamma-1)}{\gamma v p} \\ -\frac{1}{p}[1+M^2(\gamma-1)] & -\frac{M^2(\gamma-1)}{p v} \\ \frac{1}{p} & \frac{\gamma-1}{\gamma v p} \\ \frac{-M^2(\gamma-1)}{p} & \frac{-(\gamma-1)(\gamma M^2-1)}{\gamma p v} \\ \frac{M^2(\gamma-1)+2}{p} & \frac{(\gamma M^2+1)(\gamma-1)}{\gamma p v} \end{bmatrix}\begin{bmatrix} JB \\ EJ \end{bmatrix} \tag{1}$$

(a) For subsonic and supersonic generator operation, $M^2 \lessgtr 1$ and $JB>0$ while $EJ<0$. Eq. 1a gives $[J=\sigma(E+vB)]$.

$$\frac{\rho'}{\rho} = \frac{-J}{1-M^2}\left[\frac{1}{pv}\right]\left[Bv+\frac{(\gamma-1)E}{\gamma}\right] = \frac{-1}{(1-M^2)pv}\left(\frac{J^2}{\sigma}-\frac{EJ}{\gamma}\right) \lessgtr 0 \tag{2}$$

Eq. 1b can be written as

$$\frac{p'}{p} = \frac{-J}{(1-M^2)pv}\left[vB+M^2(\gamma-1)(vB+E)\right] = \frac{-1}{(1-M^2)pv}\left[vJB+(\gamma-1)\frac{M^2J^2}{\sigma}\right] \lessgtr 0 \tag{3}$$

Prob. 9.17.3 (cont.)

Except for sign, Eq. 1c is the same as Eq. 1a, so

$$\frac{v'}{v} \gtrless 0 \tag{4}$$

Eq. 1d is

$$\frac{T'}{T} = \frac{-(\gamma-1)J}{pv(1-M^2)}\left[M^2 vB + \frac{(\gamma M^2-1)}{\gamma}E\right]$$
$$= \frac{-(\gamma-1)}{pv(1-M^2)}\left(\frac{M^2J^2}{\sigma} - \frac{EJ}{\gamma}\right) \lessgtr 0 \tag{5}$$

and finally, Eq. 1e is

$$\frac{(M^2)'}{M^2} = \frac{J}{(1-M^2)pv}\left\{Bv\left[M^2(\gamma-1)+2\right] + E\frac{(\gamma M^2+1)(\gamma-1)}{\gamma}\right\}$$

$$= \frac{J}{(1-M^2)pv}\left\{(Bv+E)(\gamma-1)M^2 + 2Bv + E\frac{(\gamma-1)}{\gamma}\right\}$$

$$= \frac{J}{(1-M^2)pv}\left\{(Bv+E)(\gamma-1)M^2 + 2(Bv+E) - \frac{E(\gamma+1)}{\gamma}\right\} \tag{6}$$

$$= \frac{1}{(1-M^2)pv}\left\{\frac{J^2}{\sigma}\left[(\gamma-1)M^2+2\right] - \frac{JE(\gamma+1)}{\gamma}\right\} \gtrless 0$$

With $JB > 0$, the force is retarding the flow and it "might be expected" that the gas would slow down and that the ma density would increase. What has been found is that for subsonic flow, the velocity increases while the mass density pressure and temperature decrease. From Eq. (6), it also follows that the Mach number decreases. That is, as the gas velocity goes up and the sonic velocity goes down (the temperature goes down) the critical sonic condition $M^2 = 1$ is approached.

For supersonic flow, all conditions are reversed. The velocity decreases with increasing z while the pressure, density and temperature

Prob. 9.17.3 (cont.)

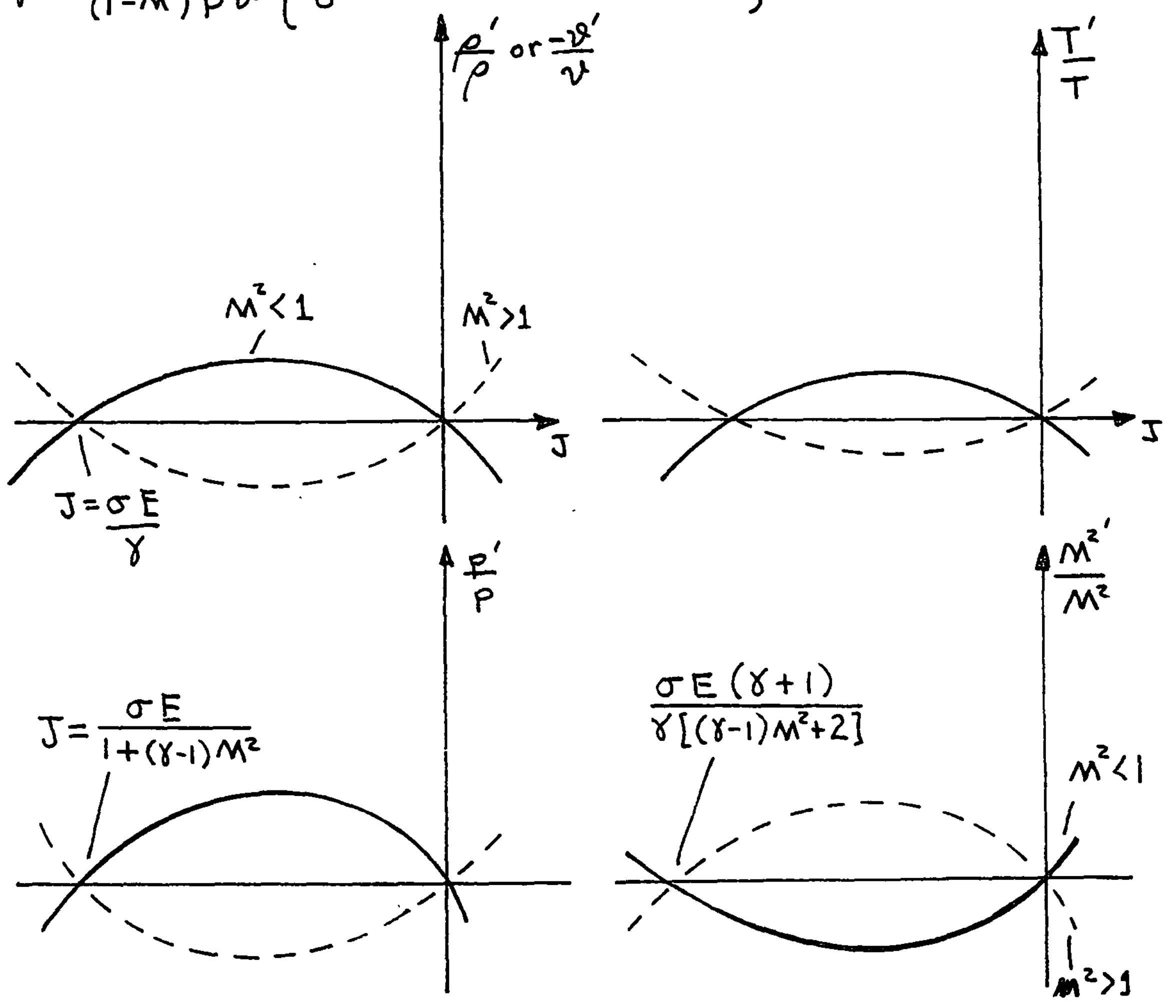
increase. However, because the Mach number is now decreasing with increasing z, the flow again approaches the critical sonic condition.

(b) In the "accelerator" mode, $EJ > 0$ and $JB < 0$. For the discussion, take B as positive so that $J < 0$, which means that

$$E + vB < 0 \Rightarrow E < -vB \tag{7}$$

Note that this means that EJ is automatically greater than zero. Note that this leaves unclear the signs of the right-hand sides of Eqs. 2-6. Consider a section of the channel where the voltage is uniformly distributed with z. Then E is constant and the dependence on J of the right-hand sides of Eqs. 2-6 can be sketched as shown in the figure. In sketching Eq. 3, it is necessary to recognize that $vB = \frac{J}{\sigma} - E$ so that Eq. 3 is also

$$\frac{p'}{p} = \frac{-1}{(1-M^2)\,pv}\left\{\frac{J^2}{\sigma}\left[1 + (\gamma-1)M^2\right] - JE\right\} \tag{8}$$



Prob. 9.17.3 (cont.)

By way of illustrating the significance of these sketches, consider the dependence of T'/T on $\mathcal{J}$. If at some location in the duct $\frac{\sigma E}{\gamma M^2} < \mathcal{J} < 0$, then the temperature is increasing with z if the flow is subsonic and decreasing if it is supersonic. The oppösite is true if $\mathcal{J} < \sigma E/\gamma M^2$. (Remember that E is negative.)

<u>Prob. 9.18.1</u> The mechanical equation of state is Eq. (d) of Table 9.15.1

$$p = \rho R T \tag{1}$$

The objective is now to eliminate ϑ, Φ and p from Eqs. 9.18.21 and 9.18.22. Substitution of the former into the latter gives

$$c_p \frac{dT}{dz} - \frac{1}{\rho}\frac{dp}{dz} = 0 \tag{2}$$

Now, with T eliminated by use of Eq. 1, this becomes

$$\frac{c_p}{R}\frac{d}{dz}\left(\frac{p}{\rho}\right) - \frac{1}{\rho}\frac{dp}{dz} = 0 \tag{3}$$

Because $R = c_p - c_v$, (Eq. 7.22.13) and $\gamma \equiv c_p/c_v$ so $c_p/R = \gamma/(\gamma-1)$, it follows that Eq. 3 can be written as

$$\frac{dp}{p} = \gamma \frac{d\rho}{\rho} \tag{4}$$

Integration from the "d" state to the state of interest gives the first of the desired expressions

$$\ln\left(\frac{p}{p_d}\right) = \gamma \ln\left(\frac{\rho}{\rho_d}\right) \Rightarrow \frac{p}{p_d} = \left(\frac{\rho}{\rho_d}\right)^{\gamma} \tag{5}$$

The second relation is simply a statement of Eq. 1 divided by p_d on the left and $\rho_d R T_d$ on the right.

Prob. 9.18.2 Because the channel is designed to make the temperature constant, it follows from the mechanical equation of state (Eq. 9.18.13) that

$$p = \rho R T \Rightarrow \frac{p}{p_d} = \frac{\rho}{\rho_d} \tag{1}$$

At the same time, it has been shown that the transition is adiabatic, so Eq. 9.18.23 holds.

$$\frac{p}{p_d} = \left(\frac{\rho}{\rho_d}\right)^{\gamma} ; \gamma \neq 1 \tag{2}$$

Thus, it follows that both the temperature and mass density must also be constant

$$p = p_d \quad ; \quad \rho = \rho_d \tag{3}$$

In turn, Eq. 9.18.10, which expresses mass conservation, becomes

$$v A = v_d A_d \tag{4}$$

and Eq. 9.18.20 can be used to show that the charge density is constant

$$\rho_f = \frac{I}{A_d v_d} \tag{5}$$

So, with the relation $E = -d\Phi/dz$, Eq. 9.18.9 is (Gauss' Law)

$$-\frac{d}{dz}\left(A\frac{d\Phi}{dz}\right) = \frac{A}{\epsilon_o}\frac{I}{A_d v_d} \tag{6}$$

In view of the isothermal condition, Eq. 9.18.22 requires that

$$\frac{1}{2}v^2 + \frac{I}{\rho_d A_d v_d}\Phi = \frac{1}{2}v_d^2 + \frac{I}{\rho_d A_d v_d}\Phi_d \tag{7}$$

The required relation of the velocity to the area is gotten from Eq. 3.

$$v = v_d \frac{A_d}{A} \tag{8}$$

and substitution of this relation into Eq. 7 gives the required expression for Φ in terms of the area.

$$\Phi = \frac{1}{2}v_d^2\left(1 - \frac{A_d^2}{A^2}\right)\frac{\rho_d A_d v_d}{I} + \Phi_d \tag{9}$$

Substitution of this expression into Eq. 5 gives the differential equation for the area dependence on z that must be used to secure a constant temperature.

$$\frac{d}{dz^2}A^{-1} - k^2 A = 0 \quad ; \quad k^2 \equiv \frac{\rho_{fd} I \rho_d^2}{\epsilon_o (A_d \rho_d v_d)^3} \tag{10}$$

Prob. 9.18.2 (cont.)

Multiplication of Eq. 10 by dA^{-1}/dt results in an expression that can be written as

$$\frac{d}{dz}\left[\frac{1}{2}\left(\frac{dA^{-1}}{dt}\right)^2 - k^2 \ln A^{-1}\right] = 0 \tag{11}$$

(Note that this approach is motivated by a similar one taken in dealing with potential-well motions.) To evaluate the constant of integration for Eq. 10, note from the derivative of Eq. 9 that E is proportional to dA^{-1}/dz

$$E = -\frac{v_d^3 \rho_d A_d^3}{I} A^{-1} \frac{dA^{-1}}{dz} \tag{12}$$

Thus, conditions at the outlet are

$$A = A_d \;;\; \frac{dA^{-1}}{dz} = 0 \quad \text{at } z = l \tag{13}$$

and Eq. 12 becomes

$$\frac{1}{2}\left(\frac{dA^{-1}}{dz}\right)^2 - k^2 \ln A^{-1} = -k^2 \ln A_d^{-1} \tag{14}$$

The second integration proceeds by writing Eq. 14 as

$$\frac{dA^{-1}}{dz} = \pm\sqrt{2k^2 \ln\left(\frac{A^{-1}}{A_d}\right)} \tag{15}$$

and introducing as a new parameter

$$x^2 = \ln\left(\frac{A^{-1}}{A_d^{-1}}\right) \Rightarrow d(A^{-1}) = 2\,x\,A_d^{-1} e^{x^2} dx \tag{16}$$

Then, Eq. 15 is

$$\pm\frac{\sqrt{2}}{k A_d}\int_x^0 e^{x^2} dx = \int_z^l dz = l - z \tag{17}$$

This expression can be written as (choosing the - sign)

$$F(x)\, e^{x^2} = (l - z)\left(\frac{\rho_{fd}^2}{2\epsilon_o \rho_d v_d^2}\right)^{1/2} \tag{18}$$

Prob. 9.18.2 (cont.)

where
$$F(x) = e^{-x^2}\int_0^x e^{x^2}dx \tag{19}$$

and Eq. 5 has been used to write $\rho_{fd} = I/A_d v_d$.

Eq. 12 and Eq. 14 evaluated at the entrance give

$$\left(\frac{I}{v_d^3 \rho_d A_d^3}\right)^2 \frac{1}{2}E_o^2 A_o^2 = k^2 \ln\left(\frac{A_d}{A_o}\right) \tag{20}$$

while from Eq. 4 $v_o A_o = v_d A_d$. Because $\rho_d = \rho_o$ this expression therefore becomes the desired one.

$$\frac{1}{2}\frac{\epsilon E_o^2}{\rho_o v_o^2} = \ln\left(\frac{A_d}{A_o}\right) \tag{21}$$

Finally, the terminal voltage follows from Eq. 9 as

$$V = \Phi_d - \Phi_o = \frac{1}{2}v_d^2\left[\left(\frac{A_d}{A_o}\right)^2 - 1\right]\frac{\rho_d A_d v_d}{I} \tag{22}$$

Thus, the electrical power out is

$$VI = \frac{1}{2}v_d^3 \rho_d A_d\left[\left(\frac{A_d}{A_o}\right)^2 - 1\right] \tag{23}$$

The area ratio A_d/A_o follows from Eq. 20 and can be substituted into Eq. 22, written using the facts that $\rho_d = \rho_o$, $v_d = v_o A_o/A_d$ as $[r \equiv (\epsilon E_o^2/2)/(\rho_o v_o^2/2)$ so that $(A_d/A_o)^2 = \exp r]$,

$$VI = v_o A_o \frac{1}{2}\rho_o v_o^2\left(\frac{A_o}{A_d}\right)^2\left[\left(\frac{A_d}{A_o}\right)^2 - 1\right] = v_o A_o \frac{\left(\frac{1}{2}\epsilon_o E_o^2\right)}{r}\left(1 - e^{-r}\right) \tag{24}$$

Thus, it is clear that the maximum power that can be extracted $\left(\frac{1}{2}\epsilon_o E_o^2 \rightarrow \infty\right)$ is the kinetic power $v_o A_o\left(\frac{1}{2}\rho_o v_o^2\right)$.

Prob. 9.18.3 With the understanding that the duct geometry is given, so that ξ'/ξ is known, the electrical relations are, Eq. 9.18.8

$$\frac{d}{dz}\left[\rho_f \pi \xi^2 (bE + v) + 2\pi \sigma_s \xi E\right] = 0 \tag{1}$$

or with primes indicating derivatives,

$$\rho_f \xi^2 (bE' + v') + \rho_f 2\xi\xi' (bE + v) + \rho_f' \xi^2 (bE + v) + 2\sigma_s \xi E' + 2\sigma_s \xi' E = 0 \tag{2}$$

Eq. 9.18.9

$$\frac{d}{dz}(\xi^2 E) + \frac{2\sigma_s}{\rho_f b}\frac{d}{dz}(\xi E) = \frac{\rho_f \xi^2}{\epsilon_0} \tag{3}$$

which is

$$\xi^2 E' + 2\xi\xi' E + \frac{2\sigma_s}{\rho_f b}\xi' E + \frac{2\sigma_s}{\rho_f b}\xi E' - \frac{\rho_f \xi^2}{\epsilon_0} = 0 \tag{4}$$

The mechanical relations are

$$\frac{d}{dz}(\rho v \xi^2) = 0 \tag{5}$$

which can be written as

$$\rho v 2\xi\xi' + \rho v' \xi^2 + \rho' v \xi^2 = 0 \tag{6}$$

Eq. 9.18.11

$$\rho v v' + p' - \rho_f E = 0 \tag{7}$$

Eq. 9.18.12

$$\rho v c_p T' + \rho v^2 v' - \rho_f E(bE + v) - \frac{2\sigma_s E^2}{\xi} = 0 \tag{8}$$

and Eq. 9.18.13

$$p' - \rho R T' - \rho' R T = 0 \tag{9}$$

Although redundant, the Mach relation is

$$M^2 = \frac{v^2}{\gamma R T} \tag{10}$$

which is equivalent to

$$M^{2\prime} - \frac{2 v v'}{\gamma R T} + \frac{v^2 T'}{\gamma R T^2} = 0 \tag{11}$$

With the definition

$$Q \equiv \left[-2\left(1 + \frac{\sigma_s}{\xi \rho_f b}\right)\frac{\xi'}{\xi} + \frac{\rho_f}{\epsilon_0 E}\right] \Big/ \left[1 + \frac{2\sigma_s}{\xi \rho_f b}\right] \tag{12}$$

Eqs. 6,7,8,2,4,9 and 11 are respectively written in the orderly form

Prob. 9.18.3 (cont.)

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ \gamma M^2 & 1 & 0 & 0 & 0 & 0 & 0 \\ M^2(\gamma-1) & 0 & 1 & 0 & 0 & 0 & 0 \\ v & 0 & 0 & v & \left(b+\frac{2\sigma_s}{\rho_f \xi}\right)E & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 & 0 \\ -2 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v'/v \\ p'/p \\ T'/T \\ \left(\frac{bE}{v}+1\right)\frac{\rho_f'}{\rho_f} \\ E'/E \\ \rho'/\rho \\ M^{2\prime}/M^2 \end{bmatrix} = \begin{bmatrix} -2\xi'/\xi \\ \rho_f E/p \\ \frac{\rho_f E(\gamma-1)}{p v \gamma}\left[\left(b+\frac{2\sigma_s}{\rho_f\xi}\right)E+v\right] \\ -2\left(bE+v+\frac{\sigma_s E}{\rho_f \xi}\right)\frac{\xi'}{\xi} \\ Q \\ 0 \\ 0 \end{bmatrix} \tag{13}$$

In the inversion of these equations, the determinant of the coefficients is

$$\text{Det} = (M^2-1)(-v) \tag{14}$$

Thus, the required relations are

$$\begin{bmatrix} v'/v \\ p'/p \\ T'/T \\ \left(\frac{bE}{v}+1\right)\frac{\rho_f'}{\rho_f} \\ E'/E \\ \rho'/\rho \\ M'^2/M^2 \end{bmatrix} = \frac{1}{M^2-1}\left[A_{ij}\right] \begin{bmatrix} -2\xi'/\xi \\ \rho_f E/p \\ \frac{\rho_f E}{p v}\left(\frac{\gamma-1}{\gamma}\right)\left(\left(b+\frac{2\sigma_s}{\rho_f\xi}\right)E+v\right) \\ -2\left(bE+v+\frac{\sigma_s E}{\xi\rho_f}\right)\left(\frac{\xi'}{\xi}\right) \\ Q \end{bmatrix} \tag{15}$$

$$A_{ij} \equiv \begin{bmatrix} -1 & 1 & -1 & 0 & 0 \\ M^2\gamma & [-M^2(\gamma-1)-1] & M^2\gamma & 0 & 0 \\ M^2(\gamma-1) & -M^2(\gamma-1) & (M^2\gamma-1) & 0 & 0 \\ 1 & -1 & 1 & \frac{(1-M^2)}{v} & (1-M^2)\frac{E}{v}\left(b+\frac{2\sigma_s}{\rho_f\xi}\right) \\ 0 & 0 & 0 & 0 & M^2-1 \\ M^2 & -1 & 1 & 0 & 0 \\ -[M^2(\gamma-1)+2] & [M^2(\gamma-1)+2] & -(M^2\gamma+1) & 0 & 0 \end{bmatrix}$$

Prob. 9.18.4 In the limit of no convection, the appropriate laws represent Gauss, charge conservation and the terminal current. These are Eqs. 9.18.8, 9.18.9 and 9.18.10.

$$\frac{d}{dz}\left(\rho_f b E \pi \xi^2 + 2\pi \sigma_s \xi E\right) = 0 \tag{1}$$

$$\frac{d}{dz}(\xi^2 E) + \frac{2\sigma_s}{\rho_f b}\frac{d}{dz}(\xi E) = \frac{\rho_f \xi^2}{\epsilon_o} \tag{2}$$

$$I = \rho_f b E_o \pi \xi^2 + 2\pi \xi \sigma_s E_o = \rho_{fo} b E_o \pi \xi_o^2 + 2\pi \xi_o \sigma_s E_o \tag{3}$$

This last expression serves to determine the entrance charge density, given the terminal current I.

$$\rho_{fo} = \frac{I - 2\pi \xi_o \sigma_s E_o}{b E_o \pi \xi_o^2} \tag{4}$$

Using this expression, it is possible to evaluate the integration constant needed to integrate Eq. 1. Thus, that expression shows that

$$\rho_f = \frac{I - 2\pi \sigma_s E_o \xi}{b E_o \pi \xi^2} \tag{5}$$

Substitution of this expression (of how the charge density thins out as the channel expands) into Eq. 2 gives a differential equation for the channel radius.

$$E_o \frac{d}{dz}\xi^2 + \frac{2\sigma_s E_o^2 \pi \xi^2}{(I - 2\pi\sigma_s E_o \xi)}\frac{d\xi}{dz} = \frac{I - 2\pi\sigma_s E_o \xi}{\epsilon_o b E_o^2 \pi} \tag{6}$$

This expression can be written so as to make it clear that it can be integrated.

$$\int_1^{\underline{\xi}} \left[\frac{2\underline{\xi}}{1-\Sigma\underline{\xi}} + \frac{\Sigma \underline{\xi}^2}{(1-\Sigma\underline{\xi})^2}\right] d\underline{\xi} = \underline{z} \tag{7}$$

where

$$\Sigma \equiv 2\pi \sigma_s E_o \xi_o / I \quad ; \quad \underline{\xi} \equiv \xi/\xi_o$$

$$\ell_o \equiv \epsilon_o b E_o^2 \pi \xi_o / I \quad ; \quad \underline{z} \equiv z/\ell_o$$

Thus, integration from the entrance, where $z = 0$ and $\xi = \xi_o$, gives

Prob. 9.18.4 (cont.)

$$\frac{1}{\Sigma^2}\left\{4\left[(1-\Sigma\xi)-(1-\Sigma)\right]-\frac{1}{2}\left[(1-\Sigma\xi)^2-(1-\Sigma)^2\right]\right. \\ \left. -3\left[\ln(1-\Sigma\xi)-\ln(1-\Sigma)\right]\right\} = z \qquad (8)$$

Given a normalized radius $\underline{\xi}$, this expression can be used to find the associated normalized position $\underline{z}$, with the normalized wall conductivity, Σ, as a dimensionless parameter.

Prob. 9.19.1 It is clear from the energy equation, Eq. 9.16.2, that because the velocity decreases (as it by definition does in a diffuser), then the temperature must increase. The temperature is related to the pressure by the mechanical equation of state, Eq. (d) of Table 9.15.1.

$$p = \rho R T \Rightarrow \frac{p}{p_o} = \frac{\rho}{\rho_o}\frac{T}{T_o} \qquad (1)$$

In the diffuser, the transition is also adiabatic, so Eq. 9.16.3 also applies

$$\frac{p}{p_o} = \left(\frac{\rho}{\rho_o}\right)^{\gamma} \qquad (2)$$

These equations can be combined to eliminate the mass density.

$$\left(\frac{p}{p_o}\right)^{(\gamma-1)/\gamma} = \frac{T}{T_o} \qquad (3)$$

Because $\gamma > 1$, it follows that because the temperature increases, so does the pressure.

Prob. 9.19.2 The fundamental equation representing components in the cycle is Eq. 9.19.7

$$\int_V \bar{E}\cdot\bar{J}\,dV = \oint_S \rho\bar{v}\left(H_T + \frac{1}{2}\bar{v}\cdot\bar{v}\right)\cdot\bar{n}\,da \tag{1}$$

In the heat-exchanger the gas is raised in temperature and entropy as it passes from $i \to f$. Here, the electrical power input represented by the left side of Eq. 1 is replaced by a thermal power input. Thus, with the understanding that the vaporized water leaves the heat exchanger at f with negligible kinetic energy,

$$\frac{\text{thermal energy input/unit time}}{\text{mass/unit time}} = H_T^f - H_T^i \tag{2}$$

In representing the turbine, it is assumed that the vapor expansion that turns the thermal energy into kinetic energy occurs within the turbine and that the gas has negligible kinetic energy as it leaves the turbine

$$-\frac{\text{turbine power output}}{\text{mass/unit time}} = \frac{-VI}{A\rho v} = H_T^g - H_T^f \tag{3}$$

Heat rejected in the condensor, $g \to h$, is taken as lost. The power required to raise the pressure of the condensed liquid, from $h \to i$, is (assuming perfect pumping efficiency)

$$\frac{\text{pump power in}}{\text{mass/unit time}} = H_T^i - H_T^g \tag{4}$$

Combining these relations and recognizing that the electrical power output is η_g times the turbine shaft power gives

$$\frac{\text{electrical power output - pumping power}}{\text{thermal power in}} = \frac{\eta_g(-H_T^g + H_T^f) - (H_T^i - H_T^g)}{H_T^f - H_T^i} \tag{5}$$

Prob. 9.19.2 (cont.)

Now, let the heat input $i \rightarrow f$ be that rejected in $e \rightarrow a$ of the MHD or EHD system of Fig. 9.19.1.

To describe the combined systems, let $\mathring{M}_T$ and $\mathring{M}_S$ represent the mass rates of flow in the topping and steam cycles respectively. The efficiency of the overall system is then

$$\eta = \frac{\text{electrical power out of topping cycle - compressor power} \quad + \text{electrical power out of steam cycle - pump power}}{\text{heat power into topping cycle}}$$

$$= \frac{\mathring{M}_T\left[(H_T^c - H_T^e) - (H_T^b - H_T^a)\right] + \mathring{M}_S\left[(H_T^f - H_T^g)\eta_g - (H_T^i - H_T^h)\right]}{\mathring{M}_T\,(H_T^c - H_T^b)} \qquad (6)$$

Because the heat rejected by the topping cycle from $e \rightarrow a$ is equal to that into the steam cycle,

$$\mathring{M}_T(H_T^e - H_T^a) = \mathring{M}_S(H_T^f - H_T^i)$$
$$\Rightarrow \frac{\mathring{M}_S}{\mathring{M}_T} = \frac{H_T^e - H_T^a}{H_T^f - H_T^i} \qquad (7)$$

and it follows that Eq. (6) can also be written as

$$\eta = \frac{\left[(H_T^c - H_T^e) - (H_T^b - H_T^a)\right] + \left[\frac{H_T^e - H_T^a}{H_T^f - H_T^i}\right]\left[(H_T^f - H_T^g)\eta_g - (H_T^i - H_T^h)\right]}{H_T^c - H_T^b} \qquad (8)$$

With the requirement that $\eta_g = 1$, and again using Eq. 7 to reintroduce $\mathring{M}_S / \mathring{M}_T$, Eq. 8 can be written as

$$\eta = \frac{\mathring{M}_T(H_T^c - H_T^b) - \mathring{M}_S(H_T^h - H_T^g)}{\mathring{M}_T(H_T^c - H_T^b)} \qquad (9)$$

This efficiency expression takes the form of Eq. 9.19.13.