Topic 19

Beam, Plate, and Shell Elements—Part I

Contents:

- **Brief review of major formulation approaches**
- **The degeneration of a three-dimensional continuum to beam and shell behavior**
- **Basic kinematic and static assumptions used**
- **Formulation of isoparametric (degenerate) general shell elements of variable thickness for large displacements and rotations**
- **Geometry and displacement interpolations**
- **The nodal director vectors**
- **Use of five or six nodal point degrees of freedom, theoretical considerations and practical use**
- **The stress-strain law in shell analysis, transformations used at shell element integration points**
- **Shell transition elements, modeling of transition zones between solids and shells, shell intersections**

Textbook: Sections 6.3.4, 6.3.5

References: The (degenerate) isoparametric shell and beam elements, including the transition elements, are presented and evaluated in

Bathe, K. J., and S. Bolourchi, "A Geometric and Material Nonlinear Plate and Shell Element," *Computers & Structures*, *11*, 23–48, 1980.

Bathe, K. J., and L. W. Ho, "Some Results in the Analysis of Thin Shell Structures," in *Nonlinear Finite Element Analysis in Structural Mechanics*, (Wunderlich, W., et al., eds.), Springer-Verlag, 1981.

Bathe, K. J., E. Dvorkin, and L. W. Ho, "Our Discrete Kirchhoff and Isoparametric Shell Elements for Nonlinear Analysis—An Assessment," *Computers & Structures*, *16*, 89–98, 1983.

References:
(continued)

The triangular flat plate/shell element is presented and also studied in

Bathe, K. J., and L. W. Ho, "A Simple and Effective Element for Analysis of General Shell Structures," *Computers & Structures, 13*, 673–681, 1981.

STRUCTURAL ELEMENTS

- Beams
- Plates
- Shells

We note that in geometrically nonlinear analysis, a plate (initially "flat shell") develops shell action, and is analyzed as a shell.

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Various solution approaches have been proposed:

- Use of general beam and shell theories that include the desired nonlinearities.
  - With the governing differential equations known, variational formulations can be derived and discretized using finite element procedures.
  - Elegant approach, but difficulties arise in finite element formulations:
    - Lack of generality
    - Large number of nodal degrees of freedom

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- Use of simple elements, but a large number of elements can model complex beam and shell structures.
  - An example is the use of 3-node triangular flat plate/membrane elements to model complex shells.
  - Coupling between membrane and bending action is only introduced at the element nodes.
  - Membrane action is not very well modeled.

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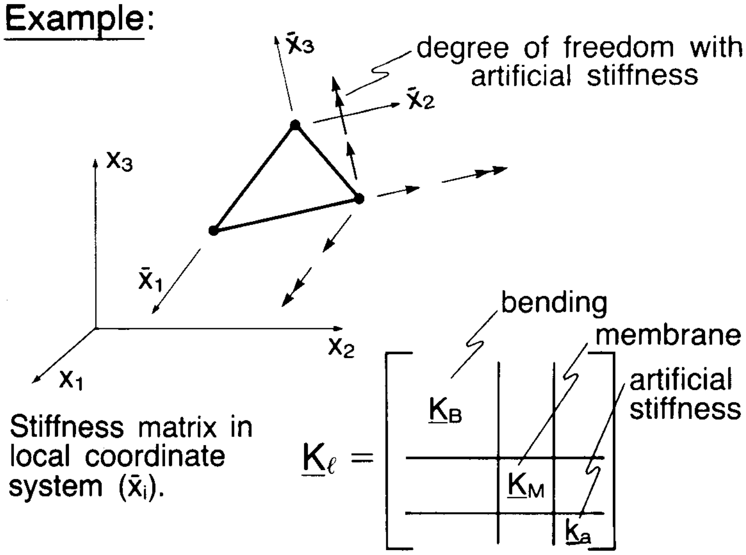
<u>Example</u>:



Stiffness matrix in local coordinate system ($\bar{x}_i$).

- Isoparametric (degenerate) beam and shell elements.
  - These are derived from the 3-D continuum mechanics equations that we discussed earlier, but the basic assumptions of beam and shell behavior are imposed.
  - The resulting elements can be used to model quite general beam and shell structures.

We will discuss this approach in some detail.

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Basic approach:

- Use the total and updated Lagrangian formulations developed earlier.

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We recall, for the T.L. formulation,

$$\int_{^0V} {}^{t+\Delta t}_{\ 0}S_{ij}\, \delta\, {}^{t+\Delta t}_{\ 0}\varepsilon_{ij}\, {}^0dV = {}^{t+\Delta t}\mathcal{R}$$

Linearization ↓

$$\int_{^0V} {}_0C_{ijrs}\, {}_0e_{rs}\, \delta_0 e_{ij}\, {}^0dV + \int_{^0V} {}^t_0S_{ij}\, \delta_0\eta_{ij}\, {}^0dV$$

$$= {}^{t+\Delta t}\mathcal{R} - \int_{^0V} {}^t_0S_{ij}\, \delta_0 e_{ij}\, {}^0dV$$

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Also, for the U.L. formulation,

$$\int_{^tV} {}^{t+\Delta t}_{\ t}S_{ij}\, \delta\, {}^{t+\Delta t}_{\ t}\varepsilon_{ij}\, {}^tdV = {}^{t+\Delta t}\mathcal{R}$$

Linearization ↓

$$\int_{^tV} {}_tC_{ijrs}\, {}_te_{rs}\, \delta_t e_{ij}\, {}^tdV + \int_{^tV} {}^t\tau_{ij}\, \delta_t\eta_{ij}\, {}^tdV$$

$$= {}^{t+\Delta t}\mathcal{R} - \int_{^tV} {}^t\tau_{ij}\, \delta_t e_{ij}\, {}^tdV$$

- Impose on these equations the basic assumptions of beam and shell action:
  1) Material particles originally on a straight line normal to the mid-surface of the beam (or shell) remain on that straight line throughout the response history.

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For beams, "plane sections initially normal to the mid-surface remain plane sections during the response history".

The effect of transverse shear deformations is included, and hence the lines initially normal to the mid-surface do not remain normal to the mid-surface during the deformations.

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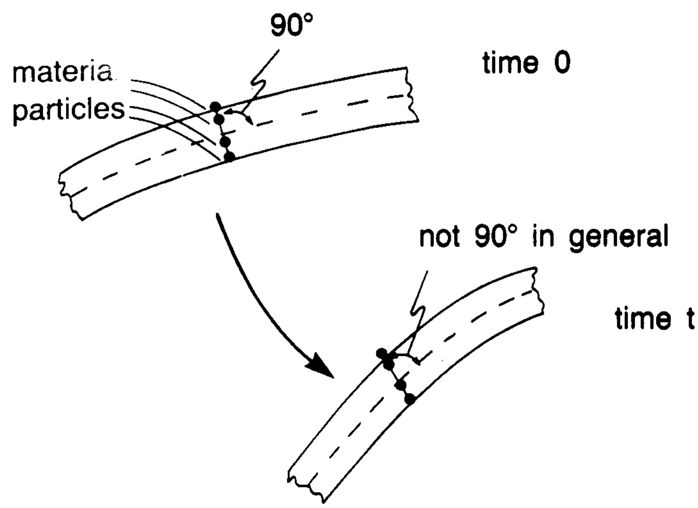

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2) The stress in the direction "normal" to the beam (or shell) mid-surface is zero throughout the response history.

   Note that here the stress along the material fiber that is initially normal to the mid-surface is considered; because of shear deformations, this material fiber does not remain exactly normal to the mid-surface.

3) The thickness of the beam (or shell) remains constant (we assume small strain conditions but allow for large displacements and rotations).

FORMULATION OF ISOPARAMETRIC (DEGENERATE) SHELL ELEMENTS

- To incorporate the geometric assumptions of "straight lines normal to the mid-surface remain straight", and of "the shell thickness remains constant" we use the appropriate geometric and displacement interpolations.
- To incorporate the condition of "zero stress normal to the mid-surface" we use the appropriate stress-strain law.

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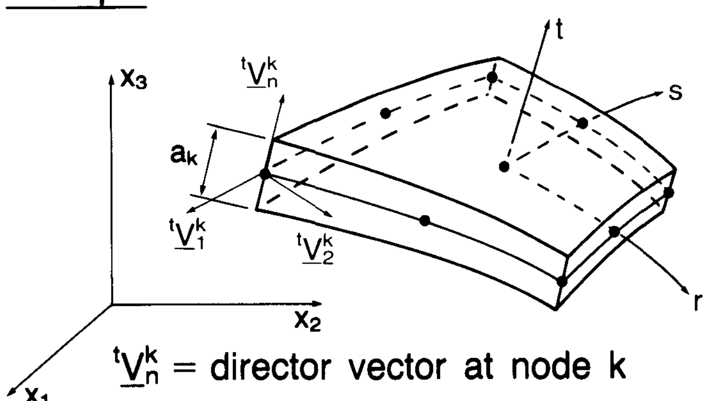
Shell element geometry

Example: 9–node element



${}^t\underline{V}_n^k$ = director vector at node k

a_k = shell thickness at node k (measured into direction of ${}^t\underline{V}_n^k$)

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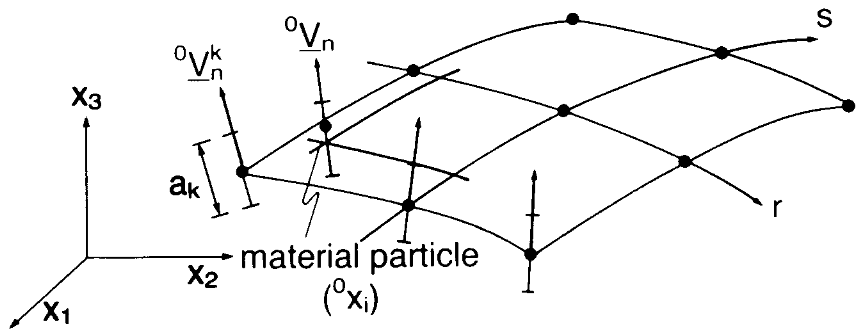
Element geometry definition:

- Input mid-surface nodal point coordinates.
- Input all nodal director vectors at time 0.
- Input thicknesses at nodes.



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- Isoparametric coordinate system (r, s, t):
  - — The coordinates r and s are measured in the mid-surface defined by the nodal point coordinates (as for a curved membrane element).
  - — The coordinate t is measured in the direction of the director vector at every point in the shell.

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Interpolation of geometry at time 0:

$$\underbrace{{}^{0}x_i}_{\substack{\text{material particle}\\ \text{with isoparametric}\\ \text{coordinates (r, s, t)}}} = \underbrace{\sum_{k=1}^{N} h_k\, {}^{0}x_i^k}_{\text{mid-surface only}} + \underbrace{\frac{t}{2}\sum_{k=1}^{N} a_k\, h_k\, {}^{0}V_{ni}^k}_{\text{effect of shell thickness}}$$

h_k = 2-D interpolation functions (as for 2-D plane stress, plane strain and axisymmetric elements)

${}^{0}x_i^k$ = nodal point coordinates

${}^{0}V_{ni}^k$ = components of ${}^{0}\underline{V}_n^k$

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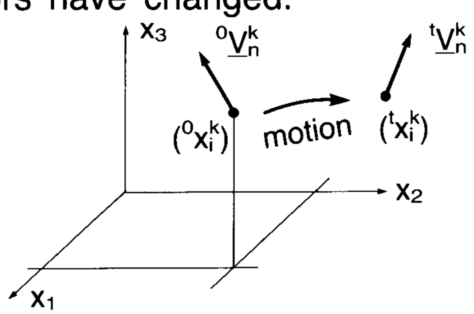
Similarly, at time t,

t-coordinate

$${}^{t}x_i = \sum_{k=1}^{N} h_k\, {}^{t}x_i^k + \frac{t}{2}\sum_{k=1}^{N} a_k\, h_k\, {}^{t}V_{ni}^k$$

The nodal point coordinates and director vectors have changed.



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To obtain the displacements of any material particle,

$$^{t}u_i = {}^{t}x_i - {}^{0}x_i$$

Hence

$$^{t}u_i = \sum_{k=1}^{N} h_k \, {}^{t}u_i^k + \frac{t}{2} \sum_{k=1}^{N} a_k \, h_k \, ({}^{t}V_{ni}^k - {}^{0}V_{ni}^k)$$

where

$^{t}u_i^k = {}^{t}x_i^k - {}^{0}x_i^k$ (disp. of nodal point k)

$^{t}V_{ni}^k - {}^{0}V_{ni}^k$ = change in direction cosines of director vector at node k

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The incremental displacements from time t to time t+Δt are, similarly, for any material particle in the shell element,

$$u_i = {}^{t+\Delta t}x_i - {}^{t}x_i$$

$$= \sum_{k=1}^{N} h_k \, u_i^k + \frac{t}{2} \sum_{k=1}^{N} a_k \, h_k \, V_{ni}^k$$

where

u_i^k = incremental nodal point displacements

$V_{ni}^k = {}^{t+\Delta t}V_{ni}^k - {}^{t}V_{ni}^k$ = incremental change in direction cosines of director vector from time t to time t+Δt

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To develop the strain-displacement transformation matrices for the T.L. and U.L. formulations, we need

— the coordinate interpolations for the material particles (${}^0x_i$, ${}^tx_i$).

— the interpolation of incremental displacements from the incremental nodal point displacements and rotations.

Hence, express the V_{ni}^k in terms of nodal point rotations.

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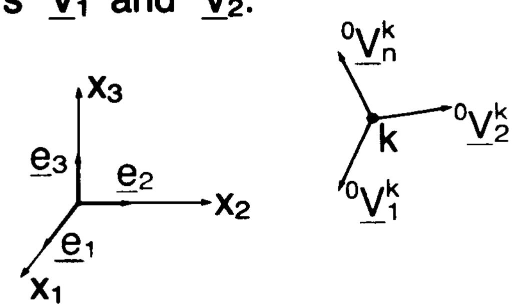
We define at each nodal point k the vectors ${}^0\underline{V}_1^k$ and ${}^0\underline{V}_2^k$:



$${}^0\underline{V}_1^k = \frac{\underline{e}_2 \times {}^0\underline{V}_n^k}{\|\underline{e}_2 \times {}^0\underline{V}_n^k\|_2} \quad , \quad {}^0\underline{V}_2^k = {}^0\underline{V}_n^k \times {}^0\underline{V}_1^k$$

The vectors ${}^0\underline{V}_1^k$, ${}^0\underline{V}_2^k$ and ${}^0\underline{V}_n^k$ are therefore mutually perpendicular.

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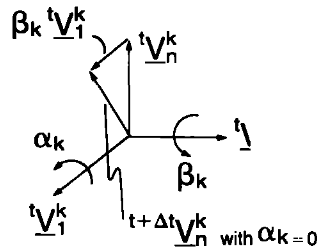
Then let α_k and β_k be the rotations about ${}^t\underline{V}_1^k$ and ${}^t\underline{V}_2^k$. We have, for small α_k, β_k,

$$\underline{V}_n^k = -\,{}^t\underline{V}_2^k\,\alpha_k + {}^t\underline{V}_1^k\,\beta_k$$



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Hence, the incremental displacements of any material point in the shell element are given in terms of incremental nodal point displacements and rotations

$$u_i = \sum_{k=1}^{N} h_k\, u_i^k + \frac{t}{2}\sum_{k=1}^{N} a_k\, h_k\,[-{}^tV_{2i}^k\,\alpha_k + {}^tV_{1i}^k\,\beta_k]$$

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Once the incremental nodal point displacements and rotations have been calculated from the solution of the finite element system equilibrium equations, we calculate the new director vectors using

$$^{t+\Delta t}\underline{V}_n^k = {}^t\underline{V}_n^k + \int_{\alpha_k,\beta_k} (-{}^\tau\underline{V}_2^k \, d\alpha_k + {}^\tau\underline{V}_1^k \, d\beta_k)$$

→ and normalize length

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Nodal point degrees of freedom:

- We have only five degrees of freedom per node:
  - — three translations in the Cartesian coordinate directions
  - — two rotations referred to the local nodal point vectors ${}^t\underline{V}_1^k$, ${}^t\underline{V}_2^k$
- The nodal point vectors ${}^t\underline{V}_1^k$, ${}^t\underline{V}_2^k$ change directions in a geometrically nonlinear solution.

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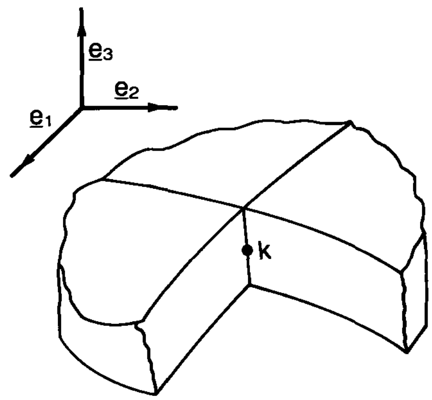

— Node k is shared by four shell elements

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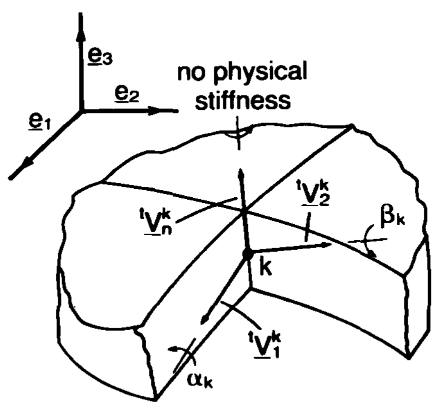
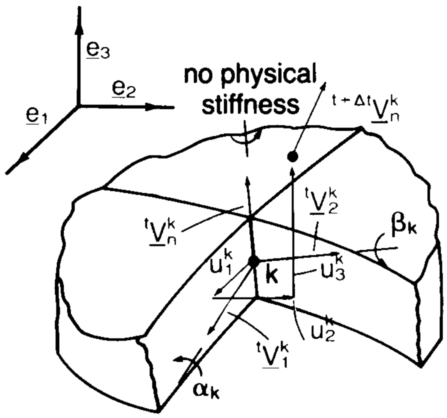


— Node k is shared by four shell elements

— One director vector ${}^t\underline{V}_n^k$ at node k

— No physical stiffness corresponding to rotation about ${}^t\underline{V}_n^k$.



— Node k is shared by four shell elements

— One director vector ${}^t\underline{V}_n^k$ at node k

— No physical stiffness corresponding to rotation about ${}^t\underline{V}_n^k$.

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- If only shell elements connect to node k, and the node is not subjected to boundary prescribed rotations, we only assign five local degrees of freedom to that node.
- We transform the two nodal rotations to the three Cartesian axes in order to
  — connect a beam element (three rotational degrees of freedom) or
  — impose a boundary rotation (other than α_k or β_k) at that node.

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- The above interpolations of ${}^0x_i$, ${}^tx_i$, u_i are employed to establish the strain-displacement transformation matrices corresponding to the Cartesian strain components, as in the analysis of 3-D solids.

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- Using the expression ${}_0e_{ij}$ derived earlier the exact linear strain-displacement matrix ${}_0^t\underline{B}_L$ is obtained.

  However, using $\frac{1}{2}\ {}_0u_{k,i}\ {}_0u_{k,j}$ to develop the nonlinear strain-displacement matrix ${}_0^t\underline{B}_{NL}$, only an approximation to the exact second-order strain-displacement rotation expression is obtained because the internal element displacements depend nonlinearly on the nodal point rotations.

  The same conclusion holds for the U.L. formulation.

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- We still need to impose the condition that the stress in the direction "normal" to the shell mid-surface is zero.

  We use the direction of the director vector as the "normal direction."

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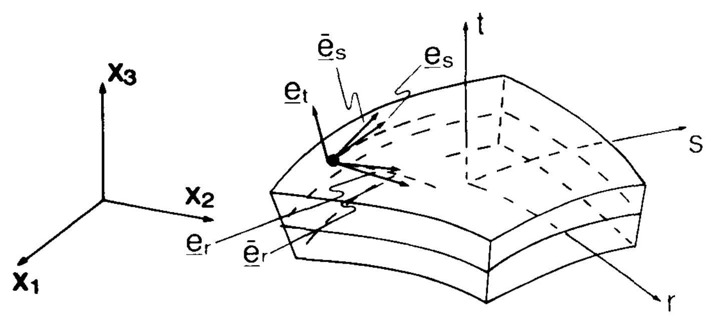

$$\bar{\underline{e}}_r = \frac{\underline{e}_s \times \underline{e}_t}{\|\underline{e}_s \times \underline{e}_t\|_2} \quad , \quad \bar{\underline{e}}_s = \underline{e}_t \times \bar{\underline{e}}_r$$

We note: $\underline{e}_r$, $\underline{e}_s$, $\underline{e}_t$ are not mutually perpendicular in general.

$\bar{\underline{e}}_r$, $\bar{\underline{e}}_s$, $\underline{e}_t$ are constructed to be mutually perpendicular.

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Then the stress-strain law used is, for a linear elastic material,

$$\underline{C}_{sh} = \underline{Q}_{sh}^T \left(\frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \\ & & & \frac{1-\nu}{2} & 0 & 0 \\ & & & & k\left(\frac{1-\nu}{2}\right) & 0 \\ \text{symmetric} & & & & & k\left(\frac{1-\nu}{2}\right) \end{bmatrix} \right) \underline{Q}_{sh}$$

k = shear correction factor

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where

$$\underline{Q}_{sh} = \begin{bmatrix} (\ell_1)^2 & (m_1)^2 & (n_1)^2 & \ell_1 m_1 & m_1 n_1 & n_1 \ell_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 2\ell_1\ell_2 & 2m_1m_2 & 2n_1n_2 & \ell_1 m_2 + \ell_2 m_1 & m_1 n_2 + m_2 n_1 & n_1 \ell_2 + n_2 \ell_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{matrix} \text{row 1} \\ \\ \text{row 4} \\ \\ \end{matrix}$$

using

$\ell_1 = \cos(\underline{e}_1, \bar{\underline{e}}_r)$ | $m_1 = \cos(\underline{e}_2, \bar{\underline{e}}_r)$ | $n_1 = \cos(\underline{e}_3, \bar{\underline{e}}_r)$

$\ell_2 = \cos(\underline{e}_1, \bar{\underline{e}}_s)$ | $m_2 = \cos(\underline{e}_2, \bar{\underline{e}}_s)$ | $n_2 = \cos(\underline{e}_3, \bar{\underline{e}}_s)$

$\ell_3 = \cos(\underline{e}_1, \underline{e}_t)$ | $m_3 = \cos(\underline{e}_2, \underline{e}_t)$ | $n_3 = \cos(\underline{e}_3, \underline{e}_t)$

- The columns and rows 1 to 3 in $\underset{\sim}{C}_{sh}$ reflect that the stress "normal" to the shell mid-surface is zero.
- The stress-strain matrix for plasticity and creep solutions is similarly obtained by calculating the stress-strain matrix as in the analysis of 3-D solids, and then imposing the condition that the stress "normal" to the mid-surface is zero.

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- Regarding the kinematic description of the shell element, transition elements can also be developed.
- Transition elements are elements with some mid-surface nodes (and associated director vectors and five degrees of freedom per node) and some top and bottom surface nodes (with three translational degrees of freedom per node). These elements are used
  — to model shell-to-solid transitions
  — to model shell intersections

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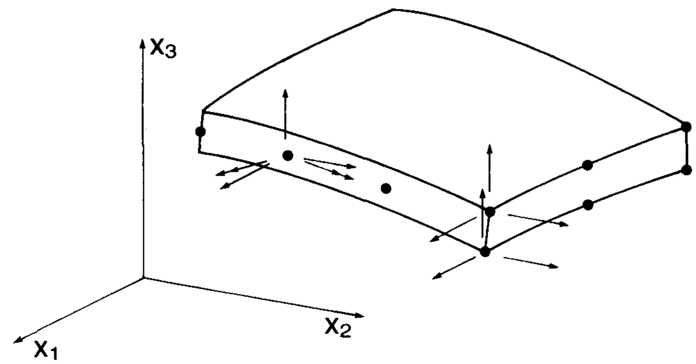

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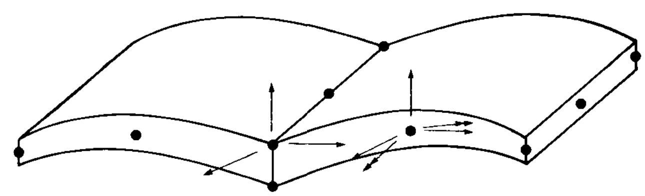
a) Shell intersection



b) Solid-shell intersection