FORMULATION OF THE DISPLACEMENT-BASED FINITE ELEMENT METHOD

LECTURE 3

58 MINUTES

LECTURE 3

General effective formulation of the displacement-based finite element method

Principle of virtual displacements

Discussion of various interpolation and element matrices

Physical explanation of derivations and equations

Direct stiffness method

Static and dynamic conditions

Imposition of boundary conditions

Example analysis of a nonuniform bar, detailed discussion of element matrices

TEXTBOOK: Sections: 4.1, 4.2.1, 4.2.2

Examples: 4.1, 4.2, 4.3, 4.4

FORMULATION OF THE DISPLACEMENT - BASED FINITE ELEMENT METHOD

- A very general formulation
- Provides the basis of almost all finite element analyses performed in practice
- The formulation is really a modern application of the Ritz/ Galerkin procedures discussed in lecture 2
- Consider static and dynamic conditions, but linear analysis

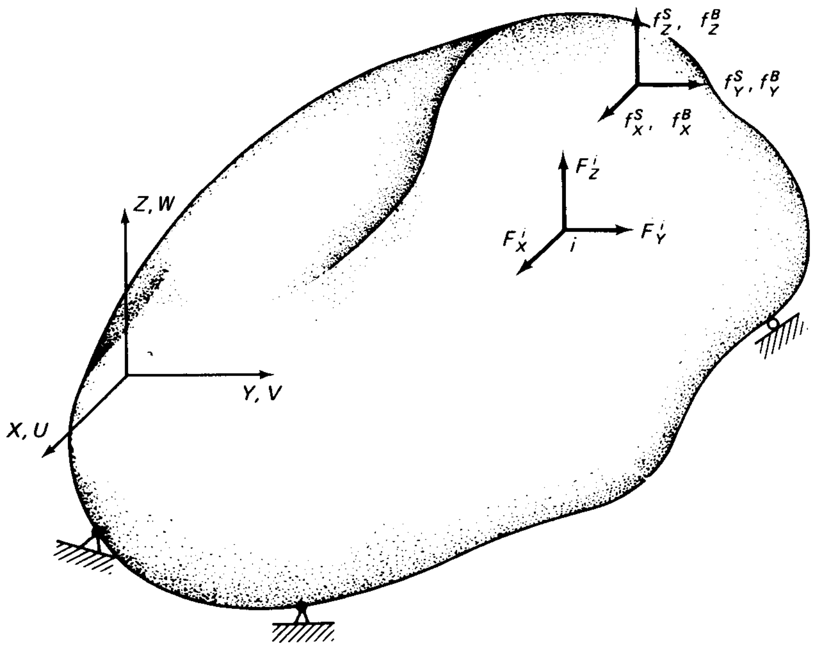


Fig. 4.2. General three-dimensional body.

The external forces are

$$\underline{f}^B = \begin{bmatrix} f_X^B \\ f_Y^B \\ f_Z^B \end{bmatrix} ; \quad \underline{f}^S = \begin{bmatrix} f_X^S \\ f_Y^S \\ f_Z^S \end{bmatrix} ; \quad \underline{F}^i = \begin{bmatrix} F_X^i \\ F_Y^i \\ F_Z^i \end{bmatrix} \qquad (4.1)$$

The displacements of the body from the unloaded configuration are denoted by U , where

$$\underline{U}^T = [U \quad V \quad W] \qquad (4.2)$$

The strains corresponding to U are,

$$\underline{\epsilon}^T = [\epsilon_{XX} \; \epsilon_{YY} \; \epsilon_{ZZ} \; \gamma_{XY} \; \gamma_{YZ} \; \gamma_{ZX}] \qquad (4.3)$$

and the stresses corresponding to ϵ are

$$\underline{\tau}^T = [\tau_{XX} \; \tau_{YY} \; \tau_{ZZ} \; \tau_{XY} \; \tau_{YZ} \; \tau_{ZX}] \qquad (4.4)$$

Principle of virtual displacements

$$\int_V \underline{\bar{\epsilon}}^T \underline{\tau} dV = \int_V \underline{\bar{U}}^T \underline{f}^B dV + \int_S \underline{\bar{U}}^{S^T} \underline{f}^S dS$$

$$+ \sum_i \underline{\bar{U}}^{i^T} \underline{F}^i \qquad (4.5)$$

where

$$\underline{\bar{U}}^T = [\bar{U} \quad \bar{V} \quad \bar{W}] \qquad (4.6)$$

$$\underline{\bar{\epsilon}}^T = [\bar{\epsilon}_{XX} \ \bar{\epsilon}_{YY} \ \bar{\epsilon}_{ZZ} \ \bar{\gamma}_{XY} \ \bar{\gamma}_{YZ} \ \bar{\gamma}_{ZX}] \qquad (4.7)$$

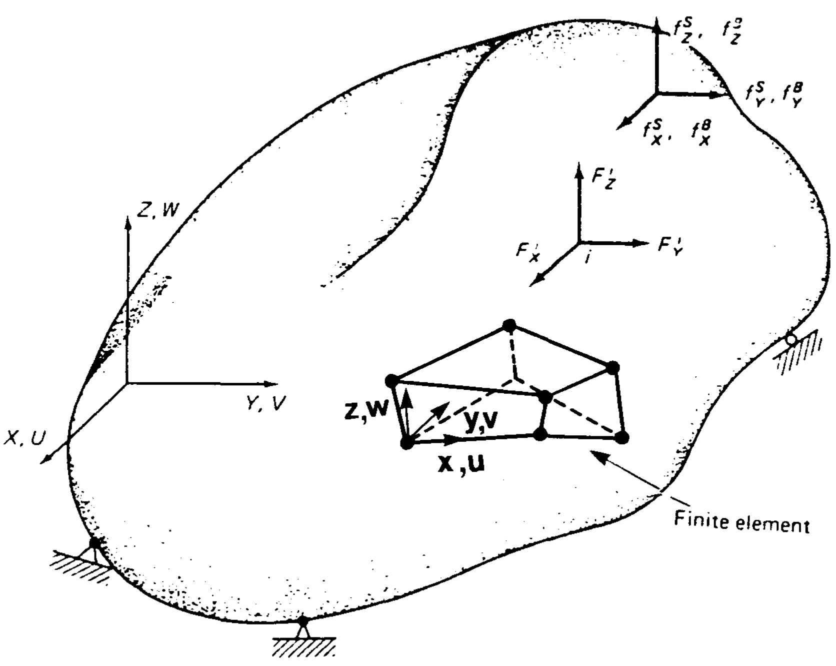
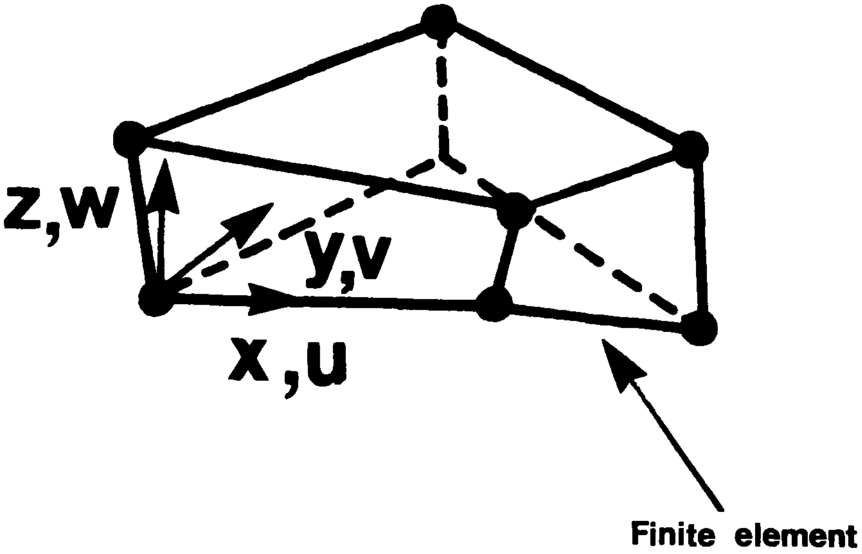


Fig. 4.2. General three-dimensional body.



For element (m) we use:

$$\underline{u}^{(m)}(x, y, z) = \underline{H}^{(m)}(x, y, z)\ \hat{\underline{U}} \qquad (4.8)$$

$$\hat{\underline{U}}^T = [U_1 V_1 W_1 \quad U_2 V_2 W_2 \ \cdots \ U_N V_N W_N] \ ;$$

$$\hat{\underline{U}}^T = [U_1 U_2 U_3 \ \cdots \ U_n] \qquad (4.9)$$

$$\underline{\epsilon}^{(m)}(x, y, z) = \underline{B}^{(m)}(x, y, z)\ \hat{\underline{U}} \qquad (4.10)$$

$$\underline{\tau}^{(m)} = \underline{C}^{(m)}\underline{\epsilon}^{(m)} + \underline{\tau}^{I(m)} \qquad (4.11)$$

Rewrite (4.5) as a sum of integrations over the elements

$$\sum_m \int_{V^{(m)}} \bar{\underline{\epsilon}}^{(m)T} \underline{\tau}^{(m)} dV^{(m)} =$$

$$\sum_m \int_{V^{(m)}} \bar{\underline{U}}^{(m)T} \underline{f}^{B(m)} dV^{(m)}$$

$$+ \sum_m \int_{S^{(m)}} \bar{\underline{U}}^{S(m)T} \underline{f}^{S(m)} dS^{(m)}$$

$$+ \sum_i \bar{\underline{U}}^{i^T} \underline{F}^i \qquad (4.12)$$

Substitute into (4.12) for the element displacements, strains, and stresses, using (4.8), to (4.10),

$$\overbrace{\cancelto{\underline{I}}{\hat{\bar{\underline{U}}}}^T \Big\{ \sum_m \int_{V^{(m)}} \underline{B}^{(m)T}}^{\bar{\underline{\epsilon}}^{(m)T}} \underbrace{\underline{C}^{(m)} \underline{B}^{(m)} dV^{(m)} \Big\} \hat{\underline{U}}}_{\underline{\tau}^{(m)}} =$$

$$\underline{\tau}^{(m)} = \underline{C}^{(m)} \; \underline{\epsilon}^{(m)}$$

$$(\bar{\underline{\epsilon}})^{(m)} = \underline{B}^{(m)} (\bar{\hat{\underline{U}}})$$

$$\cancelto{\underline{I}}{\hat{\bar{\underline{U}}}}^T \Big[\underbrace{\Big\{ \sum_m \int_{V^{(m)}} \underline{H}^{(m)T}}_{\bar{\underline{U}}^{(m)T}} \underline{f}^{B(m)} dV^{(m)} \Big\}$$

$$\bar{\underline{u}}^{(m)} = \underline{H}^{(m)} \; \bar{\hat{\underline{U}}}$$

$$+ \underbrace{\sum_m \int_{V^{(m)}} \underline{H}^{S(m)T} \underline{f}}_{\bar{\underline{U}}^{S(m)T}}{}^{S(m)} dS^{(m)} \Big\}$$

$$- \underbrace{\sum_m \int_{V^{(m)}} \underline{B}^{(m)T}}_{\bar{\underline{\epsilon}}^{(m)T}} \underline{\tau}^{I(m)} dV^{(m)} \Big\}$$

$$+ \underline{F} \Big] \qquad (4.13)$$

We obtain

$$\underline{K}\,\underline{U} = \underline{R} \qquad (4.14)$$

where

$$\underline{R} = \underline{R}_B + \underline{R}_S - \underline{R}_I + \underline{R}_C \qquad (4.15)$$

$$\underline{K} = \sum_m \int_{V^{(m)}} \underline{B}^{(m)^T} \underline{C}^{(m)} \underline{B}^{(m)} dV^{(m)} \qquad (4.16)$$

$$\underline{R}_B = \sum_m \int_{V^{(m)}} \underline{H}^{(m)^T} \underline{f}^{B^{(m)}} dV^{(m)} \qquad (4.17)$$

$$\underline{R}_S = \sum_m \int_{V^{(m)}} \underline{H}^{S\,(m)^T} \underline{f}^{S^{(m)}} dS^{(m)} \qquad (4.18)$$

$$\underline{R}_I = \sum_m \int_{V^{(m)}} \underline{B}^{(m)^T} \underline{\tau}^{I^{(m)}} dV^{(m)} \qquad (4.19)$$

$$\underline{R}_C = \underline{F} \qquad (4.20)$$

In dynamic analysis we have

$$\underline{R}_B = \sum_m \int_{V^{(m)}} \underline{H}^{(m)^T} [\underline{\tilde{f}}^{B^{(m)}} - \rho^{(m)} \underline{H}^{(m)} \underline{\ddot{U}}] dV^{(m)} \qquad (4.21)$$

$$\underline{f}^{B^{(m)}} = \underline{\tilde{f}}^{B^{(m)}} - \rho\, \underline{\ddot{u}}^{(m)}$$

$$\underline{\ddot{u}}^{(m)} = \underline{H}^{(m)}\, \underline{\ddot{U}}$$

$$\underline{M}\,\underline{\ddot{U}} + \underline{K}\,\underline{U} = \underline{R} \qquad (4.22)$$

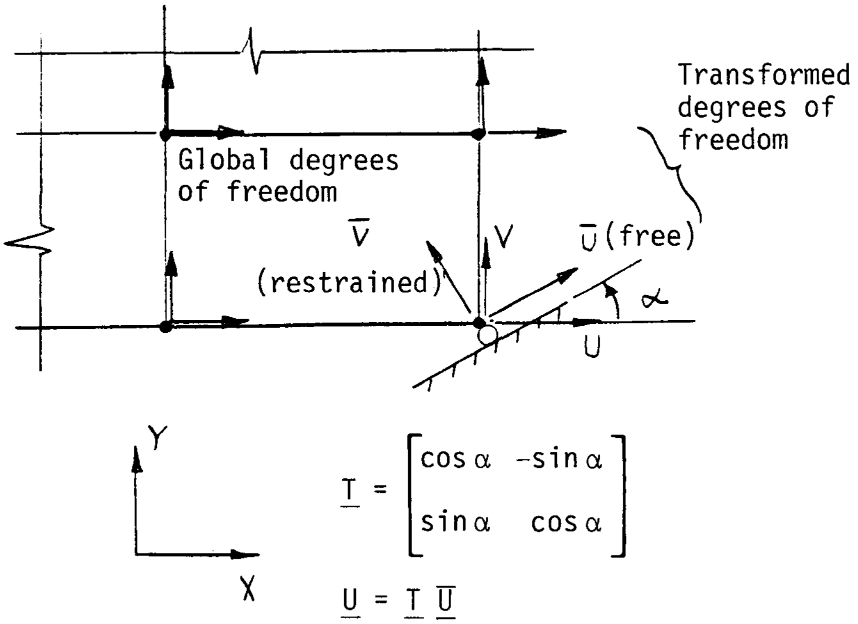
$$\underline{M} = \sum_m \int_{V^{(m)}} \rho^{(m)} \underline{H}^{(m)^T} \underline{H}^{(m)} dV^{(m)} \qquad (4.23)$$

To impose the boundary conditions, we use

$$\begin{bmatrix} \underline{M}_{aa} & \underline{M}_{ab} \\ \underline{M}_{ba} & \underline{M}_{bb} \end{bmatrix} \begin{bmatrix} \ddot{\underline{U}}_{a} \\ \ddot{\underline{U}}_{b} \end{bmatrix} + \begin{bmatrix} \underline{K}_{aa} & \underline{K}_{ab} \\ \underline{K}_{ba} & \underline{K}_{bb} \end{bmatrix} \begin{bmatrix} \underline{U}_{a} \\ \underline{U}_{b} \end{bmatrix} = \begin{bmatrix} \underline{R}_{a} \\ \underline{R}_{b} \end{bmatrix} \qquad (4.38)$$

$$\underline{M}_{aa}\,\ddot{\underline{U}}_{a} + \underline{K}_{aa}\,\underline{U}_{a} = \underline{R}_{a} - \underline{K}_{ab}\,\underline{U}_{b} - \underline{M}_{ab}\,\ddot{\underline{U}}_{b} \qquad (4.39)$$

$$\underline{R}_{b} = \underline{M}_{ba}\,\ddot{\underline{U}}_{a} + \underline{M}_{bb}\,\ddot{\underline{U}}_{b} + \underline{K}_{ba}\,\underline{U}_{a} + \underline{K}_{bb}\,\underline{U}_{b} \qquad (4.40)$$



$$\underline{T} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$\underline{U} = \underline{T}\,\bar{\underline{U}}$$

Fig. 4.10. Transformation to skew boundary conditions

For the transformation on the total degrees of freedom we use

$$\underline{U} = \underline{T}\,\underline{\bar{U}} \qquad (4.41)$$

so that

$$\underline{\bar{M}}\,\ddot{\underline{\bar{U}}} + \underline{\bar{K}}\,\underline{\bar{U}} = \underline{\bar{R}} \qquad (4.42)$$

where

$$\underline{\bar{M}} = \underline{T}^T\underline{M}\,\underline{T}\,;\ \underline{\bar{K}} = \underline{T}^T\underline{K}\,\underline{T}\ ;\ \underline{\bar{R}} = \underline{T}^T\underline{R} \qquad (4.43)$$

$$\underline{T} = \begin{array}{c|ccccccccccc|}
 & & & & i^{th}\ \text{column} & & & & j^{th} & & & \\
 & 1 & & & & & & & & & & \\
 & & \ddots & & & & & & & & & \\
 & & & 1 & & & & & & & & \\
i^{th}\ \text{row} & & & & \cos\alpha & & & & -\sin\alpha & & & \\
 & & & & & 1 & & & & & & \\
 & & & & & & \ddots & & & & & \\
 & & & & & & & 1 & & & & \\
j^{th} & & & & \sin\alpha & & & & \cos\alpha & & & \\
 & & & & & & & & & 1 & & \\
 & & & & & & & & & & \ddots & \\
 & & & & & & & & & & & 1
\end{array}$$

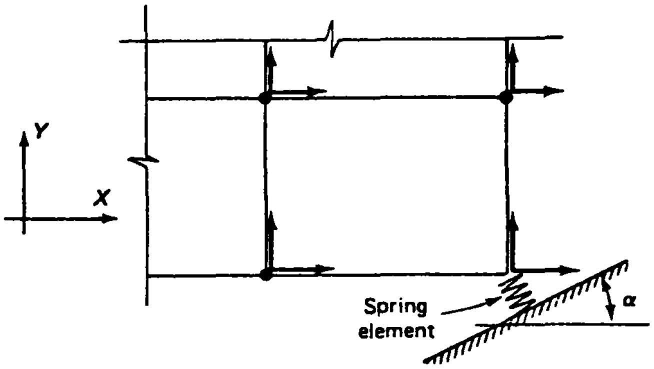


Fig. 4.11. Skew boundary condition imposed using spring element.

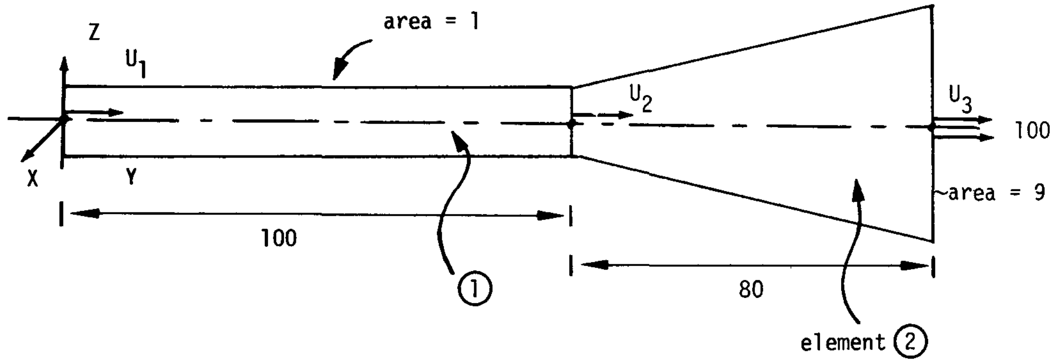
We can now also use this procedure (penalty method)
Say $U_i = b$, then the constraint equation is

$$k\,U_i = k\,b \qquad (4.44)$$

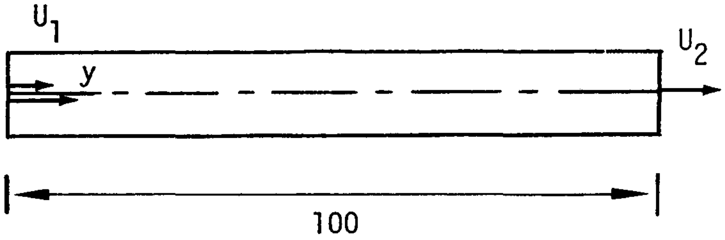
where

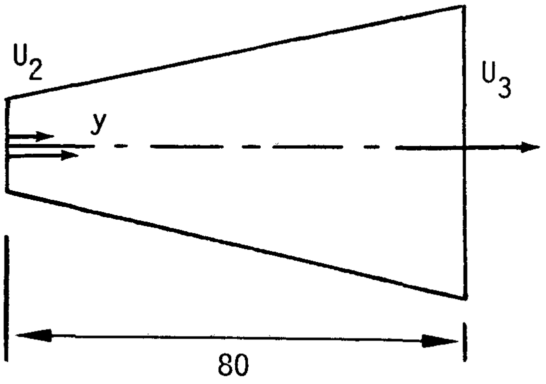
$$k \gg \bar{k}_{ii}$$

Example analysis

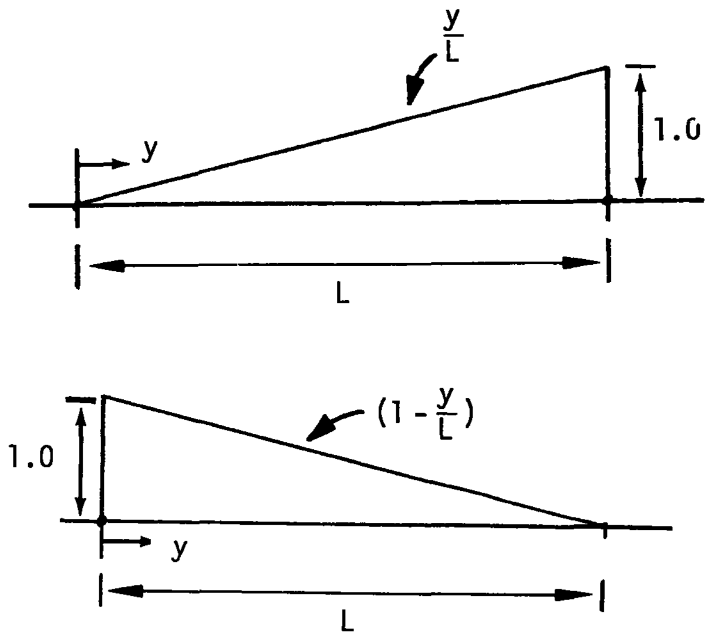


Finite elements





Element interpolation functions



Displacement and strain interpolation matrices:

$$\underline{H}^{(1)} = \left[\left(1-\frac{y}{100}\right) \quad \frac{y}{100} \quad 0 \right]$$

$$\underline{H}^{(2)} = \left[0 \quad \left(1-\frac{y}{80}\right) \quad \frac{y}{80} \right]$$

$$v^{(m)} = \underline{H}^{(m)} \underline{U}$$

$$\underline{B}^{(1)} = \left[-\frac{1}{100} \quad \frac{1}{100} \quad 0 \right]$$

$$\underline{B}^{(2)} = \left[0 \quad -\frac{1}{80} \quad \frac{1}{80} \right]$$

$$\frac{\partial v}{\partial y} = \underline{B}^{(m)} \underline{U}$$

stiffness matrix

$$\underline{K} = (1)(E)\int_0^{100} \begin{bmatrix} -\frac{1}{100} \\ \frac{1}{100} \\ 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{100} & \frac{1}{100} & 0 \end{bmatrix} dy$$

$$+ E\int_0^{80} (1+\frac{y}{40})^2 \begin{bmatrix} 0 \\ -\frac{1}{80} \\ \frac{1}{80} \end{bmatrix} \begin{bmatrix} 0 & -\frac{1}{80} & \frac{1}{80} \end{bmatrix} dy$$

Hence

$$\underline{K} = \frac{E}{100} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{13E}{240} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$= \frac{E}{240} \begin{bmatrix} 2.4 & -2.4 & 0 \\ -2.4 & 15.4 & -13 \\ 0 & -13 & 13 \end{bmatrix}$$

Similarly for $\underline{M}$, $\underline{R}_B$, and so on. Boundary conditions must still be imposed.

Resource: Finite Element Procedures for Solids and Structures
Klaus-Jürgen Bathe