Today I'm speaking about the first of the three great partial differential equations. So this one is called Laplace's equation, named after Laplace. And you see partial derivatives. So we have-- I don't have time. This equation is in steady state. I have x and y, I'm in the xy plane. And I have second derivatives in x and then y. So I'm looking for solutions to that equation.

And of course I'm given some boundary values. So time is not here. The boundary values, the boundary is in the xy plane, maybe a circle. Think about a circle in the xy plane. And on the circle, I know the solution u.

So the boundary values around the circle are given. And I have to find the temperature u inside the circle. So I know the temperature on the boundary. I let it settle down and I want to know the temperature inside. And the beauty is, it solves that basic partial differential equation.

So let's find some solutions. They might not match the boundary values, but we can use them. So u equal constant certainly solves the equation. U equal x, the second derivatives will be 0. U equal y. Here is a better one, x squared minus y squared. So the second derivative in the x direction is 2. The second derivative in the y direction is minus 2. So I have 2, minus 2, it solves the equation.

Or this one, the second derivative in x is 0. Second derivative in y is 0, those are simple solutions. But those are only a few solutions and we need an infinite sequence because we're going to match boundary conditions.

So is there a pattern here? So this is degree 0, constant. These are degree 1, linear. These are degree 2, quadratic. So I hope for two cubic ones. And then I hope for two fourth degree ones. And that's the pattern, that's the pattern. Let me find-- let me spot the cubic ones. X cubed, if I start with x cubed, of course the second x derivative is probably 6x. So I need the second y derivative to be minus 6x. And I think minus 3xy squared does it. The second derivative in y is 2 times the minus 3x is minus 6x, cancels the 6x from the second derivative there, and it works. So that fits the pattern, but what is the pattern?

Here it is. It's fantastic. I get these crazy polynomials from taking x plus iy to the different powers. Here to the first power, if n is 1, and I just have x plus iy and I take the real part, that's x. So I'll take the real part of this. The real part of this when n is 1, the real part is x.

What about when n is 2? Can you square that in your head? So we have x squared and we have i squared y squared, i squared being minus 1. So I have x squared and I have minus y squared. Look, the real part of this when n is 2, the real part of x plus iy squared, the real part is x squared minus y squared. And the imaginary part was the 2ixy. So the imaginary part that multiplies i is the 2xy. This is our pattern when n is 2.

And when n is 3, I take x plus iy cubed, and that begins with x cubed like that. And then I think that the other real part would be a minus 3xy squared. I think you should check that. And then there will be an imaginary part. Well, I think I could figure out the imaginary part as I think. Maybe something like minus-- maybe it's 3yx squared minus y cubed, something like that. That would be the real part and that would be the imaginary part when n is 3.

And wonderfully, wonderfully, it works for all powers, exponents n. So I have now sort of a pretty big family of solutions. A list, a double list, really, the real parts and the imaginary parts for every n. So I can use those to find the solution u, which I'm looking for, the temperature inside the circle.

Now of course, I have a linear equation. So if I have several solutions, I can combine them and I still have a solution. X plus 7y will be a solution. Plus 11x squared minus y squared, no problem. Plus 56 times 2xy. Those are all solutions. So I'm going to find a solution, my final solution u will be a combination of this, this, this, this, this, this, this, and all the others for higher n. That's going to be my solution. And I will need that infinite family. See, partial differential equations, we move up to infinite family of solutions instead of just a couple of null solutions.

So let me take an example. Let me take an example. We're taking the region to be a circle. So in that circle, I'm looking for the solution u of x and y. And actually in a circle, it's pretty natural to use polar coordinates. Instead of x and y inside a circle that's inconvenient in the xy plane, its equation involves x equals square root of 1 minus y squared or something, I'll switch to polar coordinates r and theta.

Well, you might say you remember we had these nice family of solutions. Is it still good in polar coordinates? Well the fact is, it's even better. So the solution of u will be the real part and the imaginary part. Now what is x plus iy in r and theta? Well, we all know x is r Cos theta plus ir sine theta. And that's r times Cos theta plus i sine theta, the one unforgettable complex Euler's formula, e to the I theta.

Now, I need its nth power. The nth power of this is wonderful. The real part and imaginary part of the nth power is r to the nth e to the in theta. That's my x plus iy to the nth. Much nicer in polar coordinates, because I can take the real part and the imaginary part right away. It's r to the nth Cos n theta and r to the nth sine n theta.

These are my solutions, my long list of solutions, to Laplace's equation. And it's some combination of those, my final thing is going to be some combination of those, some combination. Maybe coefficients a sub n. I can use these and I can use these. So maybe b sub n r to the nth sine n theta. You may wonder what I'm doing, but what I'm achieved, it's done now, is to find the general solution of Laplace's equation.

Instead of two constants that we had for an ordinary differential equation, a C1 and a C2, here I have these guys go from up to infinity. N goes up to infinity. So I have many solutions. And any combination working, so that's the general solution. That's the general solution. And I would have to match that-- now here's the final step and not simple, not always simple-- I have to match this to the boundary conditions. That's what will tell me the constants, of course. As usual, c1 and c2 came from the matching the conditions.

Now I don't have just c1 and c2, I have this infinite family of a's, infinite family of b's. And I have a lot more to match because on the boundary, here I have to match u0, which is given. So I might be given, suppose I was given u0 equal to the temperature was equal 1 on the top half. And on the bottom half, say the temperature is minus 1. That's a typical problem.

I have a circular region. The top half is held at one temperature, the lower half is held at a different temperature. I reach equilibrium. Everybody knows that along that line, probably the temperature would be 0 by symmetry. But once the temperature there halfway up, not so easy, or anywhere in there. Well, the answer is u in the middle, u of r and theta inside is given by that formula. And again, the ANs and the BNs come by matching the-- getting the right answer on the boundary.

Well, there's a big theory there how do I match these? That's called a Fourier series. That's called a Fourier series. So I'm finding the coefficients for a Fourier series, the A's and B's, that match a function around the boundary. And I could match any function, and Fourier series is another entirely separate video.

We've done the job with Laplace's equation in a circle. We've reduced the problem to a Fourier series problem. We have found the general solution. And then to match it to a specific given boundary value, that's a Fourier series problem. So I'll have to put that off to the Fourier series video. Thank you.