Unit 6: Conformal Mapping

1.6.1(L)

In the same way that one can mechanically take a derivative in the real case without having a feeling for what the answer means, one can in the complex case know that a mapping is conformal without realizing the full impact of the result. In this exercise, we hope to make it clear just what a conformal mapping really is. We begin by using the formal result without interpretation, after which we look at a specific configuration to see just what is actually meant.

a. Since $f(z) = z^2$, $f'(z) = 2z$. In particular, then, when $z = i$, $f'(z) = 2i$. Thus, in a sufficiently small neighborhood of $z = i$ in the z-plane the mapping $f(z) = z^2$ transforms a region R into the region S of w-plane in the following way: (1) If R is "centered" at i, then S [= f(R)] is centered at $f(i) = i^2 = -1$. (2) Since the magnitude of $f'(i)$ is 2, the size of S is twice that the size of R. (3) Since the argument of $f'(i)$ is 90° [i.e., in polar form $2i = (2,90°)$], S has the orientation obtained by rotating R through (+)90°.

Thus, for example, if R is a sufficiently small rectangle centered at i, then S is a rectangle centered at (w =) -1; each side of S has twice the length of the corresponding side of R; and S has the orientation obtained by rotating R through 90°. More "pictorially" if R is any sufficiently small neighborhood centered at $z = i$, "lift it up" and put it down in exactly the same orientation centered at $w = -1$. Then "blow R up" by uniformly doubling its size. Rotate the resulting region through a counter-clockwise rotation of 90°. This final configuration is S.

b. Here our main aim is to emphasize the meaning of "sufficiently small." What we are doing is taking a specific region R centered at $z = i$ and looking at its image under the mapping $f(z) = z^2$. We have described R in terms of the parameter h, where h is a positive real number, and what we shall show is that the size and shape of f(R) certainly depends on h. The technique we shall use is a review of Unit 4 in which we talked about mappings (graphs) of complex functions of a complex variable [and the mapping

1.6.1(L) continued

$f(z) = z^2$ was discussed in great detail in both the text and the exercises].

By way of review, $f(z) = z^2$ is equivalent to

$$\left.\begin{aligned} u &= x^2 - y^2 \\ v &= 2xy \end{aligned}\right\} . \tag{1}$$

Hence, the line segment $x = a$, $c \leqslant y \leqslant d$ is mapped onto

$$\left.\begin{aligned} u &= a^2 - y^2 \\ v &= 2ay \end{aligned}\right\} c \leqslant y \leqslant d \tag{2}$$

which is the parametric form of a parabola. Namely, from $v = 2ay$, we conclude that $y = \frac{v}{2a}$ and putting this result into $u = a^2 - y^2$ yields

$$u = a^2 - \left(\frac{v}{2a}\right)^2$$

or

$$4a^2u = 4a^4 - v^2$$

or

$$v^2 = 4a^2\ (a^2 - u). \tag{3}$$

From equation (3), we see that the image of the line $x = a$ is the parabola with vertex at $(a^2,0)$, v intercepts at $(0,2a^2)$ and $(0,-2a^2)$, and with the u-axis as axis of symmetry. Pictorially,

1.6.1(L) continued

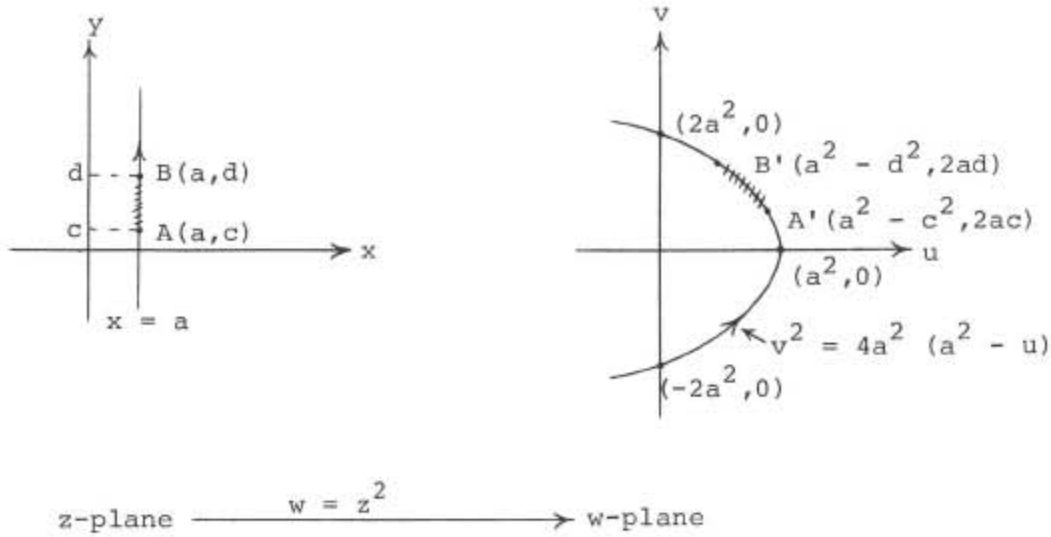


Figure 1

Since we want only the portion of $x = a$ between $c \leqslant y \leqslant d$, we see from equation (1) that the end points of our image are given by

$$\left.\begin{aligned} u &= a^2 - c^2 \\ v &= 2ac \end{aligned}\right\} \text{ and } \left\{\begin{aligned} u &= a^2 - d^2 \\ v &= 2ad \end{aligned}\right.$$

and this is also indicated in Figure 1.

In a completely analogous manner, we observe that the line $y = k$ is mapped onto the parabola defined parametrically from equation (1) by

$$u = x^2 - k^2$$

$$v = 2xk$$

That is,

$$u = \left(\frac{v}{2k}\right)^2 - k^2$$

or

1.6.1(L) continued

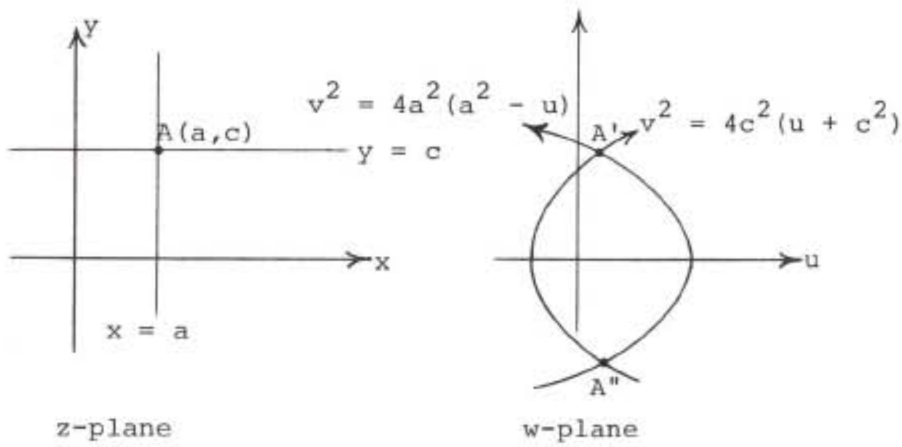
$$4k^2 u = v^2 - 4k^4$$

or

$$v^2 = 4k^2 \, (u + k^2). \tag{4}$$

Equation (4) represents the parabola in the uv-plane described by having its vertex at $(-k^2,0)$ and its v-intercepts at $(0,2k^2)$ and $(0,-2k^2)$.

In particular, the image of the lines $x = a$ and $y = c$ is given, pictorially, by



Note #1

The parabolas are orthogonal at A' by virtue of the mapping being conformal. That is, the angle between $x = a$ and $y = c$ is 90° so the image also has the same angle.

Note #2

The mapping is only <u>locally</u> 1-1. A" is the image of $(a,-c)$ and unless $c = 0$, (a,c) and $(a,-c)$ are not "sufficiently close" in the z-plane.

Figure 2

With this general discussion in mind, we turn our attention to the rectangular region R centered at $z = i$ [i.e., at $P(0,1)$] with vertices at $A(-h,1-2h)$, $B(h,1-2h)$, $C(h,1+2h)$ and $D(-h,1+2h)$.

Under the mapping $f(z) = z^2$, or,

1.6.1(L) continued

$$\begin{cases} u = x^2 - y^2 \\ v = 2xy \end{cases}$$

we have that

$$f(P) = f(i) = i^2 = -1 = P'(-1,0)$$

in the uv-plane

$$f(A) = f(-h,1-2h) = [(-h)^2 - (1-2h)^2, 2(-h)(1-2h)] =$$
$$(-1+4h - 3h^2,-2h + 4h^2) = A' \qquad (5.1)$$

$$f(B) = f(h,1-2h) = [h^2-(1-2h)^2,2h(1-2h)] = (-1+4h - 3h^2,$$
$$2h - 4h^2) = B' \qquad (5.2)$$

$$f(C) = f(h,1+2h) = [h^2-(1+2h)^2,2h(1+2h)] =$$
$$(-1-4h - 3h^2,2h + 4h^2) = C' \qquad (5.3)$$

$$f(D) = f(-h,1+2h) = [(-h)^2-(1+2h)^2,-2h(1+2h)] =$$
$$(-1-4h - 3h^2,-2h - 4h^2) = D' \qquad (5.4)$$

Thus, recalling the images of the lines x = constant and y = constant, we see pictorially that

1.6.1(L) continued

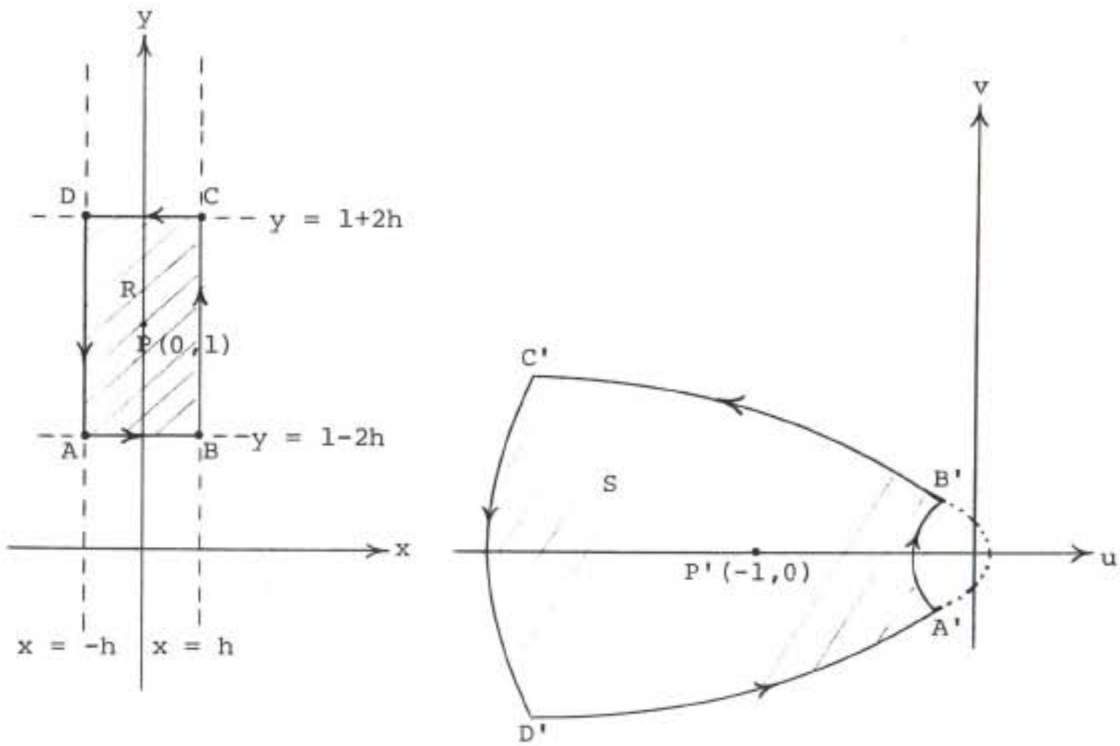


Figure 3

[Note: Our diagram in Figure 3 is based on $h = \frac{1}{4}$.]

What we hope is clear from Figure 3 is that we do not get the result predicted in part (a). The reason for this is that R, as given, is not a "sufficiently small" neighborhood of $z = i$. We do see that S has the same orientation as R; and we also perhaps suspect that S is about double the size of R; and, granted that S is a "distortion" of R, it does seem to be a 90° rotation of R.

The key to the mapping centers about the meaning of "sufficiently small" and this is the topic of the next part of this exercise.

c. Let us interpret "sufficiently small" to mean that powers of h greater than 1 are negligible. That is, we assume that R is sufficiently small so that we may assume that f behaves like a linear mapping on R. What this means more specifically is that we may return to equations 5.1, 5.2, 5.3, and 5.4 and conclude that for such a choice of h,

1.6.1(L) continued

$A' \approx (-1 + 4h, -2h) = A''$

$B' \approx (-1 + 4h, 2h) = B''$

$C' \approx (-1 - 4h, 2h) = C''$

$D' \approx (-1 - 4h, -2h) = D''$

and with this in mind, Figure 3 becomes

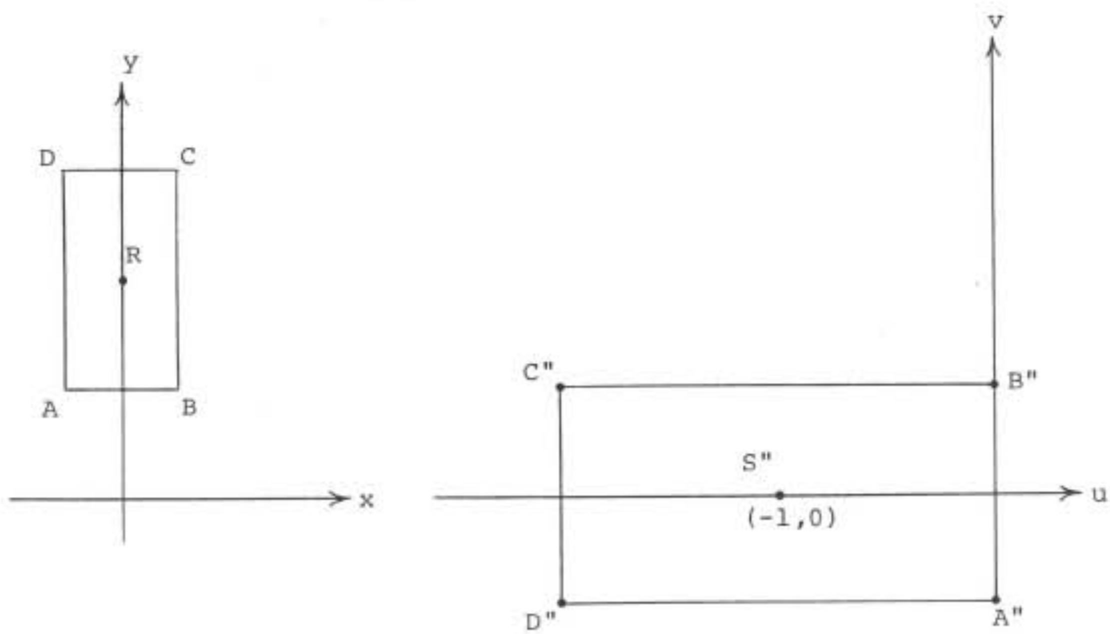


Figure 4

From Figure 4 we see that S" is obtained from R in exactly the same way as we described in part (a). That is, we lift R, center it at (-1,0) in the z-plane, double its size, and then rotate the resulting figure through an angle of (+)90°.

The problem is that unless h is sufficiently small [i.e., unless the h^2-terms are negligible in equations (5)] S" and S are quite different as regions. In other words, the fact that we may view S" rather than S as the image of R depends on h being sufficiently small, in the sense that higher powers of h can be neglected.

In fact, if we would like to dove-tail this discussion with our discussion of linear algebra in Block 4 of Part 2, we are saying that S" is obtained from R by the mapping

1.6.1(L) continued

$u = x^2 - y^2$

$v = 2xy$

only if we may assume that

$\Delta u_{tan} = 2x\Delta x - 2y\Delta y$

and

$\Delta v_{tan} = 2y\Delta x + 2x\Delta y$

are permissible substitutions for Δu and Δv.

d. If $z = 1 + i$ then $f'(1 + i) = 2(1 + i)$ so that $|f'(1 + i)| = 2\sqrt{2}$ the argument of $f'(1 + i)$ is 45°. Hence, in a sufficiently small neighborhood of $z = 1 + i$, $f(z) = z^2$ maps a region R centered at (1,1) onto a region S in the uv-plane centered at (0,2) [$= f(1,1) = (1^2-1^2, 2(1)(1)$] in the w-plane, magnifies its size by a factor of $2\sqrt{2}$ and then rotates the resulting region through 45° to yield S.

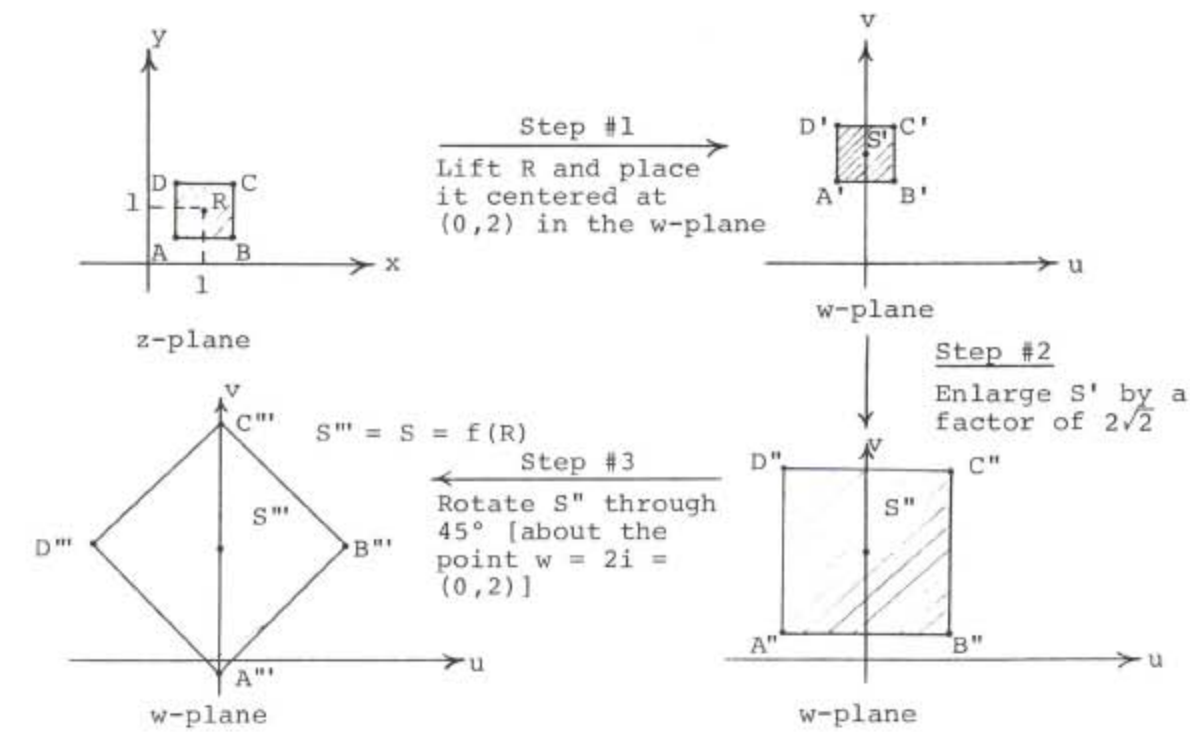


Figure 5

1.6.1(L) continued

The key point to observe here and in part (c) is that while $f(z)$ is conformal in small neighborhoods of $z = i$ and $z = 1 + i$, the behavior of f near $z = i$ is considerably different from its behavior near $z = 1 + i$. Thus, the conformal properties of f are of a local rather than a global nature.

1.6.2(L)

The main aim of this exercise is to emphasize that the conformal property of f depends not only on f being analytic but also on the fact that $f'(z)$ must be different from 0 at the point at which we want to study the transformation.

In this exercise, notice that while f' exists for each z, the fact that $f'(z) = 2z$ implies that $f'(0) = 0$. Thus, there is no reason to expect f to be conformal in any neighborhood of $z = 0$.

One way of showing that f is not conformal here is in terms of the example chosen in this exercise. We have that z^2 has an argument double that of z. Hence, the line $\theta = c°$ is mapped into the line $\theta = 2c°$. On the other hand, if the ray $\theta = c°$ is rotated through 90°, we have that the new ray is given in polar form by $\theta = c° + 90°$. Hence, under the mapping $f(z) = z^2$ this ray is mapped into the ray whose argument is twice the angle $c° + 90°$. That is, it is mapped into the ray, $\theta = 2c° + 180°$.

Pictorially, we have

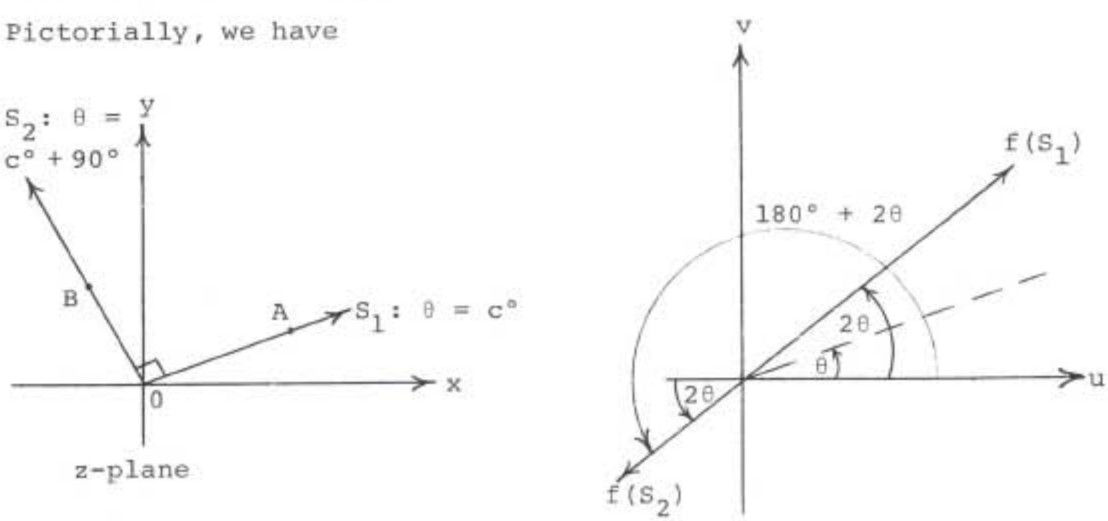


Figure 1

1.6.2(L) continued

From Figure 1, we see at a glance that while the lines S_1 and S_2 meet at right angles their images meet in a straight angle. Since the angle between S_1 and S_2 is not preserved under the mapping $z \to z^2$, we have that $f(z) = z^2$ is not conformal in any neighborhood of $z = 0$ (since any neighborhood of $z = 0$ must contain a portion of both S_1 and S_2).

Just as an aside, notice that we could have used the more familiar Cartesian coordinates to obtain the same result (even though the use of polar coordinates in this example was more conducive toward seeing how $f(z) = z^2$ rotates any line through the origin).

For example, had we used the equation of the line S_1 in the form $y = mx$ (where in this case $m = \tan c°$) with x non-negative (to indicate that our ray was in the first quadrant) we would have obtained

$$u = x^2 - (mx)^2$$

$$v = 2x(mx)$$

or

$$u = x^2(1 - m^2)$$

$$v = x^2(2m).$$

Therefore,

$$\frac{v}{u} = \frac{2m}{1 - m^2}$$

which is 2 arctan m [i.e., $\tan 2\theta = \dfrac{2 \tan \theta}{1 - \tan^2\theta}$ so that if $\tan \theta = m$, $\tan 2\theta = \dfrac{2m}{1 - m^2}$].

We thus see that if the lines are $y = mx$ and $y = -\frac{1}{m}x$ (i.e., the lines are perpendicular means their slopes are negative reciprocals) then they map into a straight angle. In fact, $y = -\frac{1}{m}x$ implies that its image is

1.6.2(L) continued

$$\frac{v}{u} = \frac{2\left(-\frac{1}{m}\right)}{1 - \left(\frac{1}{m}\right)^2} = \frac{-\frac{2}{m}}{\frac{m^2 - 1}{m^2}} = \frac{-2m}{m^2 - 1} = \frac{2m}{1 - m^2}$$

which agrees with the image of $y = mx$.

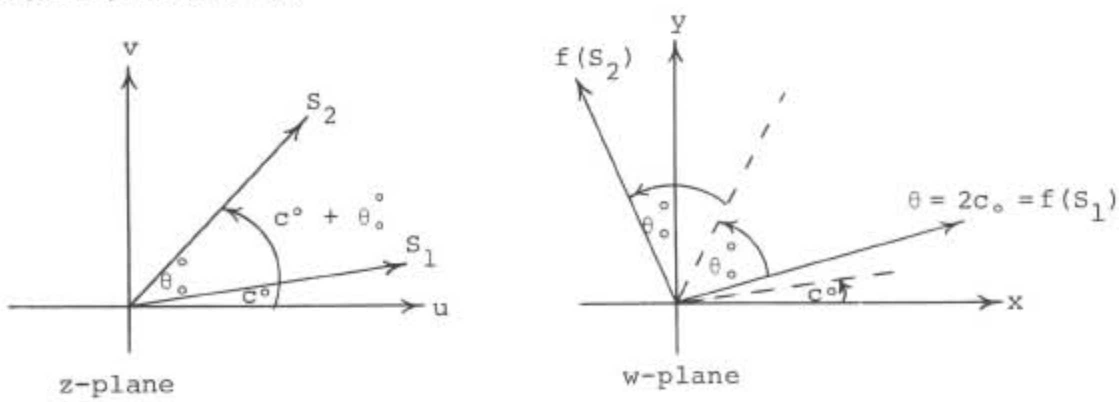
Notice that we choose the lines to be at right angles only for "dramatic impact." Had the lines met at an angle equal to $\theta_\circ^\circ$ then the image of the lines would have met an angle equal to $2\theta_\circ^\circ$. Namely,

$$\left.\begin{array}{l} \theta = c_\circ^\circ \xrightarrow{f} \theta = 2c_\circ^\circ \\ \\ \theta = c_\circ^\circ + \theta_\circ^\circ \xrightarrow{f} \theta = 2c_\circ^\circ + 2\theta_\circ^\circ \end{array}\right\}$$

The angle between these rays is $(2c_\circ^\circ + 2\theta_\circ^\circ) - 2c_\circ^\circ = 2\theta_\circ^\circ$

In summary if S_1 and S_2 are lines through the origin, then $f(S_1)$ and $f(S_2)$ are also lines through the origin but the angle formed by $f(S_1)$ and $f(S_2)$ is twice the angle formed by S_1 and S_2.

Again pictorially,



1.6.3(L)

a. The main aim of this exercise is to emphasize a subtle point about conformal mappings. Let us take it on faith that there is a one-to-one correspondence between conformal mappings and analytic functions (in neighborhoods of points at which the derivative is not equal to zero). [We have proved one part of this result,

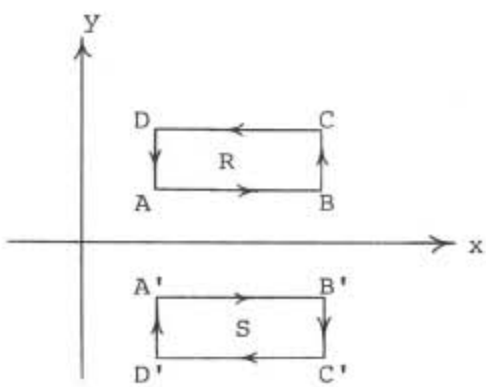
1.6.3(L) continued

namely, that if f is analytic and $f'(z_0) \neq 0$ then f is conformal in sufficiently small neighborhoods of $z = z_0$; but we shall not bother to prove the converse in our course.]

Now we saw in the lecture of the previous unit that if $f(z) = \bar{z}$ then f was not analytic. Hence, f is not conformal. BUT in Unit 4, in our discussion of various complex functions of a complex variable, we showed that the mapping $f(z) = \bar{z}$ is equivalent to a reflection about the x-axis. Clearly then, if R is any (connected) region in the xy-plane (z-plane) its image S with respect to f is congruent to R since it is merely the reflection of R with respect to the x-axis. Thus, R and S must have the same size and shape, so that, in particular, it appears that the mapping would have to be conformal.

The point that we would like to emphasize here is that "conformal" means, in addition to preserving shape, that sense is also preserved.

For example, let R be as in Figure 1. Then S is the reflection of R about the x-axis. That is,



[Notice that we have, in effect, superimposed the uv-plane onto the xy-plane in this example. This should cause no confusion since the regions R and S are non-overlapping.]

Figure 1

Notice however, that the sense of R is the opposite of the sense of f(R) = S. Namely, to enclose R so that we pass it on our left, we traverse its boundary ABCD in the counter-clockwise sense; but the image of this path encloses S in the clockwise direction.

1.6.3(L) continued

b. Suppose $f(z) = \bar{z}$. Then if $f(z_1) = f(z_2)$ it follows that

$$\bar{z}_1 = \bar{z}_2$$

and from this we may conclude that $z_1 = z_2$.

In other words, f is 1-1 since $f(z_1) = f(z_2)$ if and only if $z_1 = z_2$. [We could have deduced geometrically by observing that no two distinct (different) points in the plane can have the same image with respect to reflection about the x-axis.]

Thus, $f(z) = \bar{z}$ is not only locally 1-1 but it is globally 1-1 as well.

This proves since f is not conformal, that not all 1-1 maps are conformal.* What is true, however, is that if f is analytic and $f'(z_0) \neq 0$ [so that f is conformal in a sufficiently small neighborhood of $z = z_0$] then f is also 1-1 in a sufficiently small neighborhood of $z = z_0$. This was shown in our lecture and is repeated as part (c) of this exercise.

c. Let us utilize our previous knowledge about the Jacobian. Suppose $f = u + iv$ where $u = u(x,y)$, $v = v(x,y)$ are continuously differentiable. Then the mapping defined by f in a sufficiently small neighborhood of (x_0,y_0) is 1-1, provided that $\frac{\partial(u,v)}{\partial(x,y)} \neq 0$; that is, if

$$\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \neq 0.$$

This is turn means that

$$u_xv_y - u_yv_x \neq 0. \qquad (1)$$

Now suppose f is analytic. Then by the Cauchy-Riemann conditions

*Clearly, as we shall show in the next exercises, many other 1-1 maps are not conformal. In particular, every non-singular linear mapping of the xy-plane into the uv-plane is 1-1 but only a "few" of these are conformal. We chose to work with $f(z) = \bar{z}$ since it was already being used in part (a) of this exercise.

1.6.3(L) continued

$$u_x = v_y \text{ and } u_y = -v_x \tag{2}$$

and

$$f'(z) = u_x + iv_x. \tag{3}$$

From (3) we see that

$$|f'(z)|^2 = u_x{}^2 + v_x{}^2 \tag{4}$$

and by (2), we see that (4) is equivalent to

$$|f'(z)|^2 = (u_x)(u_x) + (v_x)(v_x)$$

$$= (u_x)(v_y) + (-u_y)v_x$$

$$= u_x v_y - u_y v_x. \tag{5}$$

Combining (5) with (1) allows us to conclude that f is 1-1 in a neighborhood of (x_o, y_o) [= $z_o = x_o + iy_o$] provided $f'(z_o) \neq 0$ [i.e., $|f'(z_o)|^2 = 0 \leftrightarrow f'(z_o) = 0$] - and this is the desired result.

Note

This result says that if f is analytic then f is 1-1 in a sufficiently small neighborhood of $z = z_o$, except possibly if $f'(z_o) = 0$. Notice that this result is completely analogous to the real variable case in which f is 1-1 in a sufficiently small neighborhood of $x = x_o$ provided $f'(x_o) \neq 0$.

1.6.4(L)

a. In Unit 4 we showed that the mapping f defined by

$$f(z) = az + b \tag{1}$$

where a and b are complex numbers, and $a \neq 0$; was conformal - even

1.6.4(L) continued

though we didn't say so in those words. Namely, we showed that az mapped z into the complex number (point) whose argument was the argument of z + arg of a, and whose magnitude was |a| times that of the magnitude of z. Thus, as long as $a \neq 0$, the mapping $g(z) = az$ rotated the z-plane through the constant angle θ_0 and multiplied each point by r_0, where (r_0, θ_0) is the polar form of the complex number, a. Then adding b $(= b_1 + ib_2)$ onto az simply translated az by the constant amount $\vec{b} = b_1\vec{i} + b_2\vec{j}$. In more "pictorial" language, if R is any region in the z-plane centered at $z = z_0$ f lifts R into the w-plane centered at $w = w_0 = az_0 + b$; magnifies it by a factor of |a|; rotates it through an angle equal to arg(a); and then translates (shifts) this region an amount $\vec{b}$. Clearly, such a chain of transformations must be conformal (except, of course, if the magnification factor is 0).

In this exercise, we want to obtain this same result using the idea of derivatives. The point is that from (1) we may conclude that

$$f'(z) = a \neq 0. \tag{2}$$

Thus, from (2) we see that f is analytic and f' is never zero. Therefore, f is conformal. In fact, from (2) we see that the magnification and rotation is constant; i.e., independent of z. In other words, just as in the real case, f is "uniformly conformal" when f is linear. That is, the effect of f in this case is the same for the entire plane.

b. If we go back to (1) and write a and b as $a_1 + ia_2$ and $b_1 + ib_2$, respectively, we obtain

$$\begin{aligned} f(z) &= (a_1 + ia_2)(x + iy) + (b_1 + ib_2) \\ &= (a_1x - a_2y) + i(a_2x + a_1y) + b_1 + ib_2 \\ &= (a_1x - a_2y + b_1) + i(a_2x + a_1y + b_2). \end{aligned} \tag{3}$$

Writing f in the form u + iv, we see that f is equivalent to the real mapping defined by

1.6.4(L) continued

$$\left.\begin{aligned} u &= a_1x - a_2y + b_1 \\ v &= a_2x + a_1y + b_2 \end{aligned}\right\} \qquad (4)$$

Hence, (4) denotes the conformal mapping which rotates the plane through the angle arctan $\frac{a_2}{a_1}$*, increases the distance of a point to the origin by a factor $\sqrt{a_1^2 + a_2^2}$ and then translates the resulting configuration through a displacement $\vec{b} = b_1 + ib_2$.

c. The mapping defined by

$$\left.\begin{aligned} u &= 3x - 2y + 5 \\ v &= 2x + 3y + 12 \end{aligned}\right\} \qquad (5)$$

is equivalent to the mapping of the z-plane into the w-plane defined by

$$\begin{aligned} f(z) &= u + iv \\ &= (3x - 2y + 5) + i(2x + 3y + 12) \\ &= (3x - 2y) + i(2x + 3y) + 5 + 12i \\ &= (3x + i3y) + i2x - 2y + 5 + 12i \\ &= 3(x + iy) + 2i(x + iy) + 5 + 12i \\ &= 3z + 2iz + 5 + 12i \\ &= (3 + 2i)z + (5 + 12i), \end{aligned} \qquad (6)$$

so that $f'(z) = 3 + 2i$.

Notice that we could deduce that $f'(z) = 3 + 2i$ without first constructing f. Namely, we see at once from (5) that $u_x = v_y$ and

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*This is defined unless $a_1 = a_2 = 0$, but the fact that $a = a_1 + ia_2$ and $a \neq 0$ guarantees that not both a_1 and a_2 can equal 0.

1.6.4(L) continued

$u_y = -v_x$. Hence, we may conclude that $f(z) = u + iv$ is analytic. We also know that in this case $f'(z)$ is given by $u_x + iv_x$. Thus, again from (5), we conclude that $f'(z) = 3 + 2i$ (since $u_x = 3$ and $v_x = 2$).

In any event, the mapping (5) rotates the xy-plane through an angle equal to $\arctan \frac{2}{3}$ [i.e., the argument of $3 + 2i$], increases the distance of each point from the origin by a factor of $\sqrt{13}$ (i.e., $|3 + 2i|$), and finally translate the "new" configuration through a displacement $\vec{b} = 5\vec{i} + 12\vec{j}$.

1.6.5(L)

a. If

$$\left.\begin{aligned} u &= 3x + 2y \\ v &= x + y \end{aligned}\right\} \quad (1)$$

then the induced complex function of a complex variable is

$$f(z) = u + iv$$

$$= (3x + 2y) + i(x + y). \quad (2)$$

To get (2) into a form which allows us to express $f(z)$ explicitly in terms of z we try to rewrite the right side of (2) in a form which emphasizes $x \pm iy$. To this end,

$$\begin{aligned}(3x + 2y) + i(x + y) &= 3x + 2y + ix + iy \\ &= (x + iy) + 2x + 2y + ix \\ &= (x + iy) + 2x + y + y + ix \\ &= (x + iy) + 2x + y + i(x - iy). \quad (3)\end{aligned}$$

With $z = x + iy$, we have $\bar{z} = x - iy$. Hence,

1.6.5(L) continued

$z + \overline{z} = 2x$*

and

$z - \overline{z} = i2y,$

so that

$y = \dfrac{z - \overline{z}}{2i}$*.

Putting these results into (3) and combining this with (1), we obtain

$$f(z) = z + (z + \overline{z}) + \frac{z - \overline{z}}{2i} + i\overline{z}$$

$$= 2z + \overline{z} + \frac{z}{2i} - \frac{\overline{z}}{2i} + i\overline{z}$$

$$= (2 + \frac{1}{2i})z + (1 - \frac{1}{2i} + i)\overline{z}$$

$$= \frac{(4i + 1)z + (2i - 1 - 2)\overline{z}}{2i}$$

$$= \frac{-2i[(4i + 1)z + (2i - 3)\overline{z}]}{4}$$

$$= \frac{(8 - 2i)z + (4 + 6i)\overline{z}}{4}$$

or

$$f(z) = \left[\frac{4 - i}{2}\right] z + \left[\frac{2 + 3i}{2}\right] \overline{z}. \qquad (4)$$

*Actually, this is just another way of saying that Re $z(= x) = \dfrac{z + \overline{z}}{2}$ and Im $z(= y) = \dfrac{z - \overline{z}}{2i}$.

1.6.5(L) continued

Check

$$\left(\frac{4-i}{2}\right)z + \left(\frac{2+3i}{2}\right)\overline{z} = \frac{1}{2}[(4-i)(x+iy) + (2+3i)(x-iy)]$$

$$= \frac{1}{2}[4x + y + i(-x + 4y) + (2x + 3y) + i(3x - 2y)]$$

$$= \frac{1}{2}[6x + 4y + i(-x + 4y + 3x - 2y)]$$

$$= \frac{1}{2}[6x + 4y + i(2x + 2y)]$$

$$= \underbrace{(3x + 2y)}_{u} + i\underbrace{(x + y)}_{v},$$

which agrees with equation (1).

b. Since $c\overline{z}$ is not differentiable (unless $c = 0$) we see that f, as defined by equation (4), is not analytic. Hence, f is not conformal.

[As a direct proof that the function f defined by equation (1) is not analytic, we need only observe that the Cauchy-Riemann conditions are not obeyed. Namely, $u_x = 3$ and $v_y = 1$; hence, $u_x \neq v_y$.]

c. If $u = 3x + 2y$, then $u_x = 3$ and $u_y = 2$. Hence, by the Cauchy-Riemann conditions, it follows that

$$v_y = u_x = 3$$

so that

$$v = 3y + g(x). \tag{5}$$

From (5)

$$v_x = g'(x)$$

but we must also have that

$$v_x = -u_y = -2.$$

1.6.5(L) continued

Hence,

$$g'(x) = -2$$

and, therefore,

$$g(x) = -2x + c. \tag{6}$$

Putting (6) into (5) yields

$$v = 3y - 2x + c \tag{7}$$

and this checks with the result of the previous exercise with $a_1 = 3$ and $a_2 = -2$.

Thus, up to translation there is only one way in which $v(x,y)$ can be chosen if

$$\left.\begin{aligned} u &= 3x + 2y \\ v &= v(x,y) \end{aligned}\right\}$$

is to be conformal. Namely,

$$v(x,y) = 3y - 2x.$$

1.6.6(L)

$$u = \frac{x}{x^2 + y^2}$$

implies

$$u_x = \frac{(x^2 + y^2) - x(2x)}{\left(x^2 + y^2\right)^2} = \frac{y^2 - x^2}{\left(x^2 + y^2\right)^2} \tag{1}$$

and

1.6.6(L) continued

$$u_y = \frac{(x^2 + y^2)0 - x(2y)}{\left(x^2 + y^2\right)^2} = \frac{-2xy}{\left(x^2 + y^2\right)^2}. \quad (2)$$

On the other hand

$$v = \frac{-y}{x^2 + y^2}$$

implies

$$v_x = \frac{(x^2 + y^2)(0) + y(2x)}{\left(x^2 + y^2\right)^2} = \frac{2xy}{\left(x^2 + y^2\right)^2} \quad (3)$$

and

$$v_y = \frac{(x^2 + y^2)(-1) + y(2y)}{\left(x^2 + y^2\right)^2} = \frac{y^2 - x^2}{\left(x^2 + y^2\right)^2}. \quad (4)$$

Looking at (1), (2), (3), and (4), we see that u_x, u_y, v_x, and v_y exist unless $x = y = 0$ and

$$\left.\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}\right\} \quad (5)$$

Equation (5) gives the Riemann-Cauchy conditions; hence, f, defined by

$$f(z) = u + iv = \frac{x}{x^2 + y^2} + i\left(\frac{-y}{x^2 + y^2}\right)$$

is analytic as long as $(x,y) \neq (0,0)$ [i.e., f is not defined at (0,0)], and since

1.6.6(L) continued

$$f'(z) = u_x + iv_x$$

$$= \frac{y^2 - x^2}{\left(x^2 + y^2\right)^2} + i\left[\frac{2xy}{\left(x^2 + y^2\right)^2}\right]$$

we have that

$$|f'(z)|^2 = \frac{\left(y^2 - x^2\right)^2 + (2xy)^2}{\left(x^2 + y^2\right)^4}$$

$$= \frac{\left(x^2 + y^2\right)^2}{\left(x^2 + y^2\right)^4};\ (x,y) \neq (0,0)$$

$$= \frac{1}{\left(x^2 + y^2\right)^2}$$

$$\neq 0.$$

Therefore, f is conformal in a sufficiently small neighborhood of any point $z = z_o = 0$.

Note

If f is defined by

$$f(z) = \frac{1}{z},\ z \neq 0$$

then

$$f'(z) = -\frac{1}{z^2} \neq 0$$

for each $z \neq 0$. Hence, f is conformal.

Now

1.6.6(L) continued

$$\frac{1}{z} = \frac{1}{x + iy}$$

$$= \frac{x - iy}{(x + iy)(x - iy)}$$

$$= \frac{x - iy}{x^2 + y^2}$$

$$= \frac{x}{x^2 + y^2} + i\left(\frac{-y}{x^2 + y^2}\right).$$

In other words, as long as $(x,y) \neq (0,0)$ $\frac{x}{x^2 + y^2}$ and $\frac{-y}{x^2 + y^2}$ are the real and imaginary parts respectively of $\frac{1}{z}$. In fact, this exercise was made up by starting with $f(z) = \frac{1}{z}$ and then rewriting f in the form u + iv.

1.6.7(L)

$$u = e^x \cos y$$

implies that

$$u_x = e^x \cos y \text{ and } u_y = -e^x \sin y. \qquad (1)$$

On the other hand

$$v = e^x \sin y$$

implies that

$$v_x = e^x \sin y \text{ and } v_y = e^x \cos y. \qquad (2)$$

Comparing (1) and (2) reveals that

$$u_x = v_y \text{ and } u_y = -v_x$$

so that

1.6.7(L) continued

$$\left.\begin{aligned} u &= e^x \cos y \\ v &= e^x \sin y \end{aligned}\right\} \qquad (3)$$

defines an analytic function

$$f(z) = u + iv$$

$$= e^x \cos y + ie^x \sin y. \qquad (4)$$

b. If f is analytic and $f = u + iv$, then

$$f' = u_x + iv_x.$$

Hence, by (3), we have

$$f'(z) = e^x \cos y + ie^x \sin y. \qquad (5)$$

Therefore,

$$|f'(z)| = \sqrt{(e^x \cos y)^2 + (e^x \sin y)^2}$$

$$= \sqrt{e^{2x}(\cos^2 y + \sin^2 y)}$$

$$= e^x \neq 0 \qquad (6)$$

for every x.

Hence, the mapping defined by (3) is conformal in a sufficiently small neighborhood of each point $z = z_o$. In fact, we see from (5) that in a neighborhood of $z = z_o = x_o + iy_o$ f rotates z by an angle equal to y (in radians) since the slope of $f'(z)$ is $\frac{\sin y}{\cos y} = \tan y$, and magnifies z by a factor of e^x.

c. $T_x = T_u u_x + T_v v_x$

$$= T_u e^x \cos y + T_v e^x \sin y$$

1.6.7(L) continued

$$T_{xx} = T_{ux}e^x \cos y + T_u e^x \cos y + T_{vx}e^x \sin y + T_v e^x \sin y$$

Hence,

$$T_{xx} = (T_{uu}u_x + T_{uv}v_x)e^x \cos y + T_u e^x \cos y + (T_{vu}u_x + T_{vv}v_x)$$
$$e^x \sin y + T_v e^x \sin y$$

$$= (T_{uu}e^x \cos y + T_{uv}e^x \sin y)e^x \cos y + T_u e^x \cos y$$
$$+ (T_{vu}e^x \cos y + T_{vv}e^x \sin y)e^x \sin y + T_v e^x \sin y$$

$$\left.\begin{aligned} &= e^{2x} \cos^2 y\, T_{uu} + 2e^{2x} \sin y \cos y\, T_{uv} + e^{2x} \sin^2 y\, T_{vv} \\ &\quad + e^x \cos y\, T_u + e^x \sin y\, T_v \end{aligned}\right\} \qquad (7)$$

Similarly,

$$T_y = T_u u_y + T_v v_y$$
$$= T_u(-e^x \sin y) + T_v e^x \cos y$$

Hence

$$T_{yy} = T_{uy}(-e^x \sin y) + T_u(-e^x \cos y) + T_{vy}e^x \cos y + T_v(-e^x \sin y)$$

$$= (T_{uu}u_y + T_{uv}v_y)(-e^x \sin y) - e^x \cos y\, T_u + (T_{vu}u_y + T_{vv}v_y)$$
$$e^x \cos y - e^x \sin y\, T_v$$

$$= [T_{uu}(-e^x \sin y) + T_{uv}(e^x \cos y)](-e^x \sin y) - e^x \cos y\, T_u$$
$$+ [T_{vu}(-e^x \sin y) + T_{vv}(e^x \cos y)]e^x \cos y - e^x \sin y\, T_v$$

In other words,

1.6.7(L) continued

$$\left.\begin{aligned} T_{yy} &= e^{2x}\sin^2 y\, T_{uu} - 2e^{2x}\sin y\cos y\, T_{uv} + e^{2x}\cos^2 y\, T_{vv} \\ &\quad -e^{x}\cos y\, T_u - e^{x}\sin y\, T_v \end{aligned}\right\} \quad (8)$$

Adding equations (7) and (8), we obtain

$$\begin{aligned} T_{xx} + T_{yy} &= e^{2x}\, T_{uu} + e^{2x}\, T_{vv} \\ &= e^{2x}\,(T_{uu} + T_{vv}) \end{aligned} \quad (9)$$

and from equation (6), we see that

$$T_{xx} + T_{yy} = |f'(z)|^2\,(T_{uu} + T_{vv})$$

which checks with the general result derived in the lecture.

In other words, if $f'(z) \neq 0$ for each $z \in R$ then
$T_{xx} + T_{yy} \equiv 0$ in $R \leftrightarrow T_{uu} + T_{vv} \equiv 0$ in $S = f(R)$.

Note

$x = x_o$ is mapped into the curve

$$\left.\begin{aligned} u &= e^{x_o}\cos y \\ v &= e^{x_o}\sin y \end{aligned}\right\}$$

or

$$u^2 + v^2 = e^{2x_o}$$

which is a circle of radius e^{x_o} centered at (0,0).

$y = y_o$ is mapped into the curve

$$\left.\begin{aligned} u &= e^{x}\cos y_o \\ v &= e^{x}\sin y_o \end{aligned}\right\}$$

1.6.7(L) continued

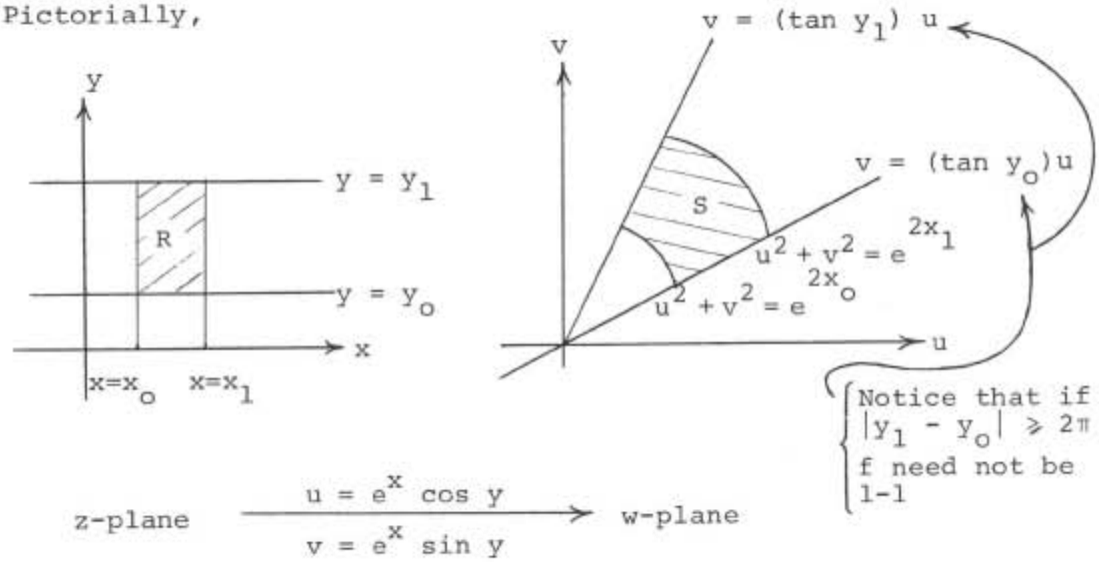
or

$$\frac{v}{u} = \frac{\sin y_o}{\cos y_o}$$

i.e.,

$$v = (\tan y_o)u$$

Pictorially,



Inversely, we may view S as being mapped onto R by f^{-1}. In other words, then, f^{-1} allows us to try to solve Laplace's equation on the rectangle R and if we can solve the problem on R, we may back-map to find the solution on S, the point being that R may be easier to handle than S.

1.6.8 (Optional)

The mapping defined by $f(z) = az + b$ where a and b are complex constants may be viewed as being a special case of the more general function defined by

$$f(z) = \frac{az + b}{cz + d} \qquad (1)$$

1.6.8 continued

where a, b, c, and d are complex numbers. In particular, if, in (1), we choose c to be zero and d to be 1, we obtain $f(z) = az + b$.

The family of functions of the type (1) in which $ad - bc \neq 0$ are known under the name of THE LINEAR GROUP. The name stems from the fact that any such function maps lines (and circles) onto lines and circles; where in this context one looks at a line as the special case of a circle with an "infinite radius."

a. With f defined by

$$f(z) = \frac{2z + i}{iz + 1},\ z \neq i \tag{1}$$

we have by the quotient rule,

$$f'(z) = \frac{(iz + 1)2 - (2z + i)i}{(iz + 1)^2}$$

$$= \frac{3}{(iz + 1)^2}. \tag{2}$$

From (2) we see that in any region R of the plane which excludes the point $z = i$, the function f is analytic and $f'(z)$ is never zero since the numerator in (2) is never 0 (i.e., it is always 3).

Thus, the mapping defined by (1) is conformal.

b. From equations (1) and (2), we have that

$$f(0) = i \tag{3}$$

and

$$f'(0) = 3. \tag{4}$$

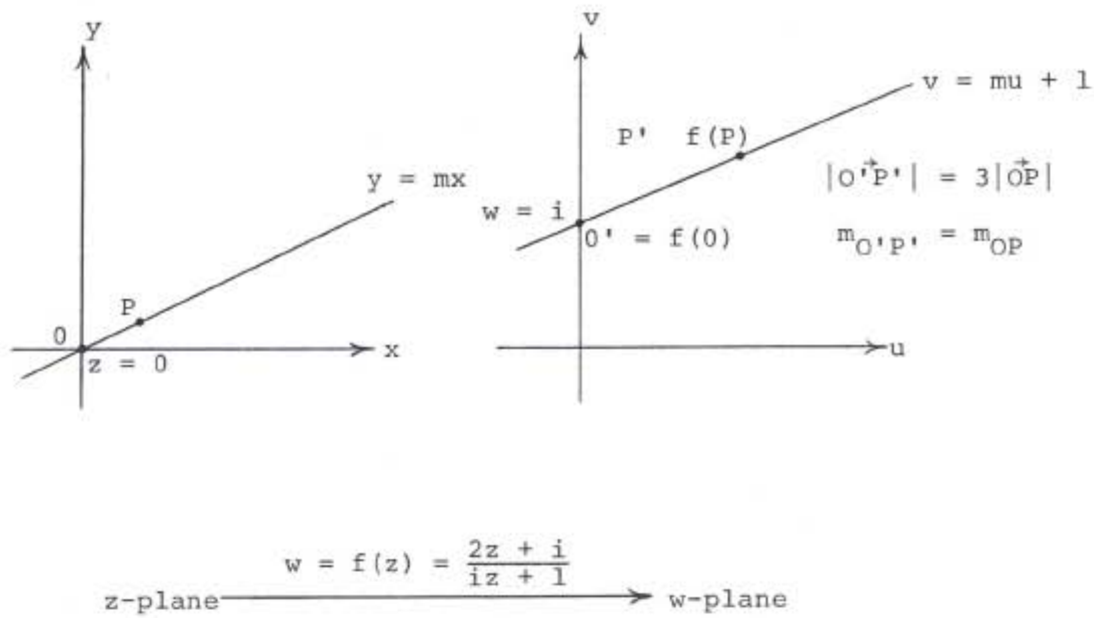
From equation (3) we may conclude that a neighborhood of $z = 0$ is mapped into a neighborhood of $w = i$.

From equation (4) we see that if z is near $z = 0$ then its image w is three times that distance from $w = i$; and since the argument of

1.6.8 continued

$f'(0)$ is $0°$ (i.e., 3 is on the positive real axis) we see that the direction (including sense) of z from $z = 0$ is the same as that of w from $w = i$.

Pictorially,



c. Here we again see the local nature of a conformal mapping. Namely, with $z = 1$, we see from equations (1) and (2) that

$$f(1) = \frac{2 + i}{i + 1}$$

$$= \frac{2 + i}{1 + i}$$

$$= \frac{(2 + i)(1 - i)}{(1 + i)(1 - i)}$$

$$= \frac{3 - i}{2} \qquad (5)$$

and

1.6.8 continued

$$f'(1) = \frac{3}{(1 + i)^2}$$

$$= \frac{3}{1 + 2i + i^2}$$

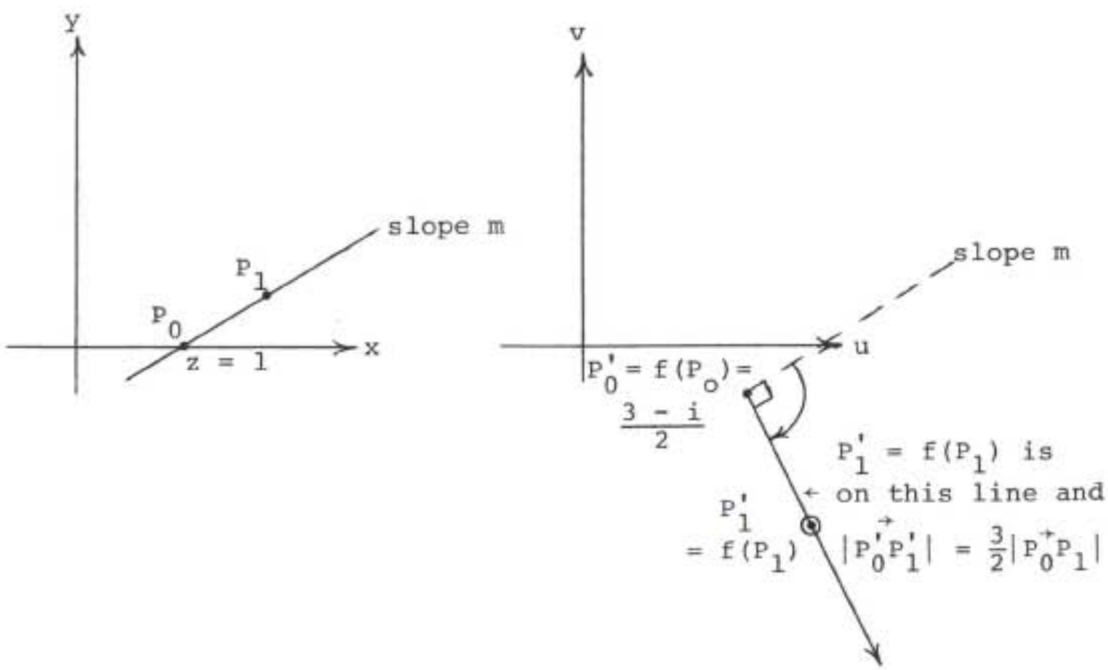
$$= \frac{3}{2i}$$

$$= \frac{3}{2}(-i). \qquad (6)$$

From (5) we have that f maps a neighborhood of $z = 1$ into a neighborhood of $w = \frac{3 - i}{2}$.

From (6) we see that for a point z near $z = 1$, its image will be $\frac{3}{2}$ that distance from $w = i$.

Finally, since $\arg f'(1) = \arg(-i) = 270° = -90°$, we see that w is rotated 90° in the clockwise direction. That is,



Notice again that f behaves very differently near $z = 1$ than it does near $z = 0$.

1.6.8 continued

d. Given that

$$f(z) = \frac{z - i}{iz + 1} \quad (z \neq i) \tag{7}$$

we see that

$$f'(z) = \frac{(iz + 1)1 - (z - i)(i)}{(iz + 1)^2}$$

$$= \frac{iz + 1 - iz + i^2}{(iz + 1)^2}$$

$$\equiv 0. \tag{8}$$

Equation (8) tells us that f(z) must be constant! In fact, with this as a hint, notice that (7) may be written in the form

$$f(z) = \frac{(z - i)}{i(z - i)} \tag{9}$$

and as long as $z \neq i$, equation (9) is equivalent to

$$f(z) = \frac{1}{i} = -i.$$

In other words,

$$f(z) = \begin{cases} -i, \text{ if } z \neq i \\ \text{undefined, or } \infty, \text{ when } z = i \end{cases}$$

In any event, f is not conformal anywhere since $f'(z) \equiv 0$.

e. More generally, if

$$f(z) = \frac{az + b}{cz + d},\ z \neq -\frac{d}{c} \tag{10}$$

then

1.6.8 continued

$$f'(z) = \frac{(cz + d)a - (az + b)c}{(cz + d)^2}$$

$$= \frac{ad - bc}{(cz + d)^2}.$$

Hence, f(z) is conformal unless ad - bc = 0 in which case f(z) is constant.

1.6.9 (Optional)

At first glance, there seems to be little relevance in the present exercise with respect to the topic of this unit. The point is, however, that the solution of this exercise gives us an excellent insight to defining "the point at infinity" in a very natural way. What we shall show is that the construction mentioned in this exercise gives us a 1-1 correspondence between the points on S, excluding P_o and the points in the xy-plane. Since the correspondence is 1-1 and onto, it is invertible. In this way, every point in the plane is identified with one and only one point on the sphere; and P_o is then viewed as the image of "infinity."

In other words, we may view the Argand Diagram as being equivalent to the sphere $x^2 + y^2 + z^2 = 1$ (the so-called Riemann Sphere), with $P_o(0,0,1)$ playing the role of the point at infinity. In this context, the point at infinity is no different from any other point on the sphere.

At any rate, with respect to the problem, let $P_1(x_1,y_1,z_1)$ be any point, other than $P_o(0,0,1)$ on S. Then the line which joins P_1 and P_o passes through (0,0,1) and has the direction of $\vec{P_oP_1} = x_1\vec{i} + y_1\vec{j} + (z_1 - 1)\vec{k}$.

Hence, the equation of this line is

$$\frac{x - 0}{x_1} = \frac{y - 0}{y_1} = \frac{z - 1}{z_1 - 1}. \qquad (1)$$

To find where this line meets the xy-plane, we merely let z = 0 in (1), in which case we obtain

1.6.9 continued

$$\frac{x}{x_1} = \frac{y}{y_1} = \frac{-1}{z_1 - 1}$$

or

$$\frac{x}{x_1} = \frac{-1}{z_1 - 1}$$

and

$$\frac{y}{y_1} = -\frac{1}{z_1 - 1}.$$

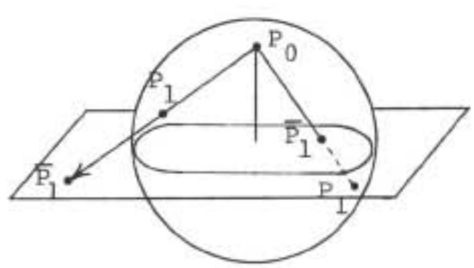
That is

$$\left. \begin{aligned} x &= \frac{x_1}{1 - z_1} \\ y &= \frac{y_1}{1 - z_1} \end{aligned} \right\} \qquad (2)$$

[Since $P_1 \neq P_o$, $z_1 \neq 1$; hence, the values of x and y, given by (2), are well-defined.]

Thus, we may match the point $P_1(x_1,y_1,z_1)$ on S with the point $\overline{P}_1\left(\frac{x_1}{1 - z_1}, \frac{y_1}{1 - z_1}\right)$ in the xy-plane.

Pictorially,



While the result is probably clear pictorially, the fact is that we can prove algebraically that the given construction maps $S - P_o$

1.6.9 continued

in a 1-1 manner onto the xy-plane. We do not wish to pursue the details here since the computation is "messy" and tends to obscure our main point that ∞ need not be "feared" if we use the present interpretation.

In the context of $f(z) = \frac{az + b}{cz + d}$ we may now view f as being defined at every point in the z-plane by letting $f(-\frac{d}{c}) = \infty$ = "the point at infinity." Of course, if we feel uncomfortable this way, we may still assume that f is defined everywhere except at $z = -\frac{d}{c}$.

1.6.10 (Optional)

a. If $f_1(z) = \frac{a_1z + b_1}{c_1z + d_1}$ and $f_2(z) = \frac{a_2z + b_2}{c_2z + d_2}$ then

$$\begin{aligned} f_1 \circ f_2(z) &= f_1[f_2(z)] \\ &= f_1\left[\frac{a_2z + b_2}{c_2z + d_2}\right] \\ &= \frac{a_1\left[\frac{a_2z + b_2}{c_2z + d_2}\right] + b_1}{c_1\left[\frac{a_2z + b_2}{c_2z + d_2}\right] + d_1} \\ &= \frac{a_1a_2z + a_1b_2 + b_1c_2z + b_1d_2}{c_1a_2z + c_1b_2 + c_2d_1z + d_1d_2} \\ &= \frac{(a_1a_2 + b_1c_2)z + a_1b_2 + b_1d_2}{(c_1a_2 + c_2d_1)z + c_1b_2 + d_1d_2}. \qquad (1) \end{aligned}$$

Equation (1) suggests the following matrix code.

If f is defined by $f(z) = \frac{az + b}{cz + d}$, let us use the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to denote f.

With this notation in mind, f_1 would be denoted by

1.6.10 continued

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \tag{2}$$

f_2 by

$$\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}. \tag{3}$$

Moreover, from (1), $f_1 \circ f_2$ is denoted by

$$\begin{pmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\ c_1a_2 + c_2d_1 & c_1b_2 + d_1d_2 \end{pmatrix} \tag{4}$$

Assuming that the rules of matrix algebra apply when our elements are complex numbers, we observe that matrix (4) is the product of (2) and (3), i.e.,

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\ c_1a_2 + d_1c_2 & c_1b_2 + d_1d_2 \end{pmatrix} \tag{5}$$

b. With $f_1(z) = \frac{z + 1}{z - 1}$ and $f_2(z) = \frac{z}{z + 2}$, we see from (5) that

$f_1 \circ f_2$

is given by

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 0 & -2 \end{pmatrix}$$

That is,

$$f_1\left(f_2(z)\right) = \frac{2z + 2}{-2} = -z - 1.$$

Similarly,

1.6.10 continued

$$f_2 \circ f_1 = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$$

Hence,

$$f_2\left(f_1(z)\right) = \frac{z+1}{3z-1}$$

c.

$$\begin{array}{r l} & \frac{a}{c} \\ cz + d & \overline{\big) az + b} \\ & \underline{az + \frac{ad}{c}} \\ & b - \frac{ad}{c} \end{array}$$

Hence,

$$\frac{az + b}{cz + d} = \frac{a}{c} + \frac{b - \frac{ad}{c}}{cz + d}^{*} \qquad (6)$$

$$= \frac{a}{c} + \frac{bc - ad}{c(cz + d)}$$

$$= \frac{a}{c} + \frac{bc - ad}{c^2(z + \frac{d}{c})}$$

$$= \left[\frac{bc - ad}{c^2}\right]\left(\frac{1}{z + \frac{d}{c}}\right) + \frac{a}{c}. \qquad (7)$$

Equation (7) now gives us an excellent insight into why mappings of the form

*If $c = 0$ then $f(z) = \frac{az + b}{d} = \frac{a}{d}z + \frac{b}{d}$ which is the linear function discussed in Exercise 1.6.4. In other words, while (6) is meaningless if $c = 0$, the case $c = 0$ is un"interesting" since in this event we have already solved the problem.

1.6.10 continued

$$f(z) = \frac{az + b}{cz + d} \ (ad - bc \neq 0)$$

carry lines and circles into lines and circles.

More explicitly, notice that (7) may be viewed as a composition of mappings each of which has the desired property. In particular,

$$z \xrightarrow{f_1} \underbrace{z + \frac{d}{c}}_{w_1} \xrightarrow{f_2} \underbrace{\frac{1}{z + \frac{d}{c}}}_{w_2 = \frac{1}{w_1}} \xrightarrow{f_3} \underbrace{\left[\frac{bc - ad}{c^2}\right]\left(\frac{1}{z + \frac{d}{c}}\right)}_{w_3} \xrightarrow{f_4} w_3 + \frac{a}{c}$$

That is, define f_1, f_2, f_3, and f_4 by

$$f_1(z) = z + \frac{d}{c},\ f_2(z) = \frac{1}{z},\ f_3(z) = \left(\frac{bc - ad}{c^2}\right)z, \text{ and } f_4(z) = z + \frac{a}{c}.$$

Then

$$f(z) = \frac{az + b}{cz + d} = f_4\left(f_3\left(f_2\left(f_1(z)\right)\right)\right)$$

or

$$f = f_4 \circ f_3 \circ f_2 \circ f_1. \tag{8}$$

We have already studied mappings of the type f_1, f_3, and f_4. In fact, f_1 and f_4 are simply translations while f_3 is a magnification $\left(\text{of magnitude } \left|\frac{bc - ad}{c^2}\right|\right)$ followed by a rotation $\theta = \arg\left(\frac{bc - ad}{c^2}\right)$. [Again, the only time we are in trouble is if $c = 0$, but we are excluding this case.]

Thus, we need only worry about the function f_2 (which is called an inversion). Let us, therefore, concentrate on what the image of $f_2(z)$ is where $f_2(z) = \frac{1}{z}$. [Again, if $c = 0$, f_2 is not needed in (8) since then f is simply given by $f(z) = \frac{az + b}{d} = \frac{a}{d}z + \frac{b}{d}$.]

1.6.10 continued

Using polar coordinates, we see that $z = (r,\theta)$ is mapped into $\frac{1}{(r,\theta)} = \frac{(1,0°)}{(r,\theta)} = (\frac{1}{r},-\theta)$. [Recall, in polar form, that when we divide complex numbers, we divide their magnitudes, and subtract their arguments.)

Hence, if R is a circle centered at the origin with radius r, $\frac{1}{r}$ is also constant. Moreover, as θ goes from 0° to 360°, $-\theta$ goes from 360° to 0° so under f_2, c is mapped onto the circle centered at the origin with radius $\frac{1}{r}$ and the opposite sense of R.

The arithmetic becomes quite a bit more awkward if c is not centered at the origin. We shall tackle this case indirectly in the next exercise, but for now we hope that you begin to get a feeling for the type of linearity possessed by $f(z) = \frac{az + b}{cz + d}$.

1.6.11 (Optional)

a. Here we show that every member of the linear group has an inverse, which is itself a member of the linear group.

In particular, let

$$w = f(z) = \frac{az + b}{cz + d}. \tag{1}$$

We may solve for z in terms of w to obtain

$$cwz + dw = az + b,$$

$$(cw - a)z = -dw + b,$$

or

$$z = \frac{-dw + b}{cw - a}. \tag{2}$$

[If $ad - bc = 0$, then $\frac{-dw + b}{cw - a}$ is a constant.]

If we reverse the roles of w and z in (2), we obtain

$$w = \frac{-dz + b}{cz - a} \tag{3}$$

1.6.11 continued

or

$$f^{-1}(z) = \frac{-dz + b}{cz - a}. \tag{4}$$

As a check of (4), let us observe from (1) and (4) that

$$f^{-1}\big(f(z)\big) = f^{-1}\left[\frac{az + b}{cz + d}\right]$$

$$= \frac{-d\left[\frac{az + b}{cz + d}\right] + b}{c\left[\frac{az + b}{cz + d}\right] - a}$$

$$= \frac{-daz - bd + bcz + bd}{caz + bc - acz - ad}$$

$$= \frac{(bc - da)z}{bc - ad},$$

or, since $ad - bc \neq 0$,

$$f^{-1}\big(f(z)\big) = z.$$

So

$$f^{-1} \circ f = \text{Identity mapping}.$$

b. There is nothing special about the points 0, 1, and ∞. In fact, we intend to show as we go along that every member of the linear group is determined as soon as we know what it does to <u>any three distinct</u> points in the z-plane.

The reasons for choosing 0, 1, and ∞ is that these three points make the arithmetic rather easy. In fact, using the notation of equation (1),

(i) $f(0) = 0 \leftrightarrow b = 0$, since $f(0) = \frac{b}{d}$

(ii) $f(1) = 1 \leftrightarrow a + b = c + d$, since $f(1) = \frac{a + b}{c + d}$

1.6.11 continued

(iii) $\dfrac{az + b}{cz + d} = \dfrac{a + \frac{b}{z}}{c + \frac{d}{z}}$; hence, $f(\infty) = \dfrac{a + \frac{b}{\infty}}{c + \frac{d}{\infty}}$

or,

$$f(\infty) = \frac{a + 0}{c + 0} = \frac{a}{c}$$

Therefore,

$$f(\infty) = \infty \leftrightarrow c = 0.$$

Now, with $b = c = 0$ [from (i) and (iii)], (ii) tells us that

$$a = d,$$

and this coupled with $b = c = 0$ implies that

$$\frac{az + b}{cz + d} = \frac{az}{a}.$$

Hence, the conditions

$$f(0) = 0,\ f(1) = 1, \text{ and } f(\infty) = \infty$$

imply that

$$f(z) = \frac{az}{a}, \qquad (5)$$

and since $ad - bc \neq 0$ and $ad - bc$ is also equal to a^2 (since $a = d$ and $b = c = 0$), it follows that $a \neq 0$. Hence, from (5), we conclude that

$$f(z) = z.$$

c. What we first show is that one such required function f is given by

1.6.11 continued

$$f(z) = \left(\frac{z_2 - z_3}{z_2 - z_1}\right)\left[\frac{z - z_1}{z - z_3}\right] \tag{6}$$

To derive (6), we observe that for $f(z_1)$ to be zero, we want $z - z_1$ to be a factor of our numerator, and to have $f(z_3) = \infty$ we want $z - z_3$ to be a factor of our denominator.

This means that $f(z)$ must be a multiple of $\frac{z - z_1}{z - z_3}$; say,

$$f(z) = k\left[\frac{z - z_1}{z - z_3}\right]. \tag{7}$$

Knowing that $f(z_2) = 1$, we put this information into (7) to deduce that

$$1 = f(z_2) = k\left[\frac{z_2 - z_1}{z_2 - z_3}\right],$$

whereupon it follows that

$$k = \frac{z_2 - z_3}{z_2 - z_1}. \tag{8}$$

Combining equations (7) and (8) establishes equation (6).

Now, to prove that f, as defined by (6), is the only such member of the linear group, suppose that g is any member of the linear group such that g also satisfies $g(z_1) = 0$, $g(z_2) = 1$ and $g(z_3) = \infty$.

By part (a) of this exercise, g^{-1} exists. Moreover,

$$\left.\begin{aligned} f\left(g^{-1}(0)\right) &= f(z_1) = 0 \\ f\left(g^{-1}(1)\right) &= f(z_2) = 1 \\ f\left(g^{-1}(\infty)\right) &= f(z_3) = \infty \end{aligned}\right\} \tag{9}$$

1.6.11 continued

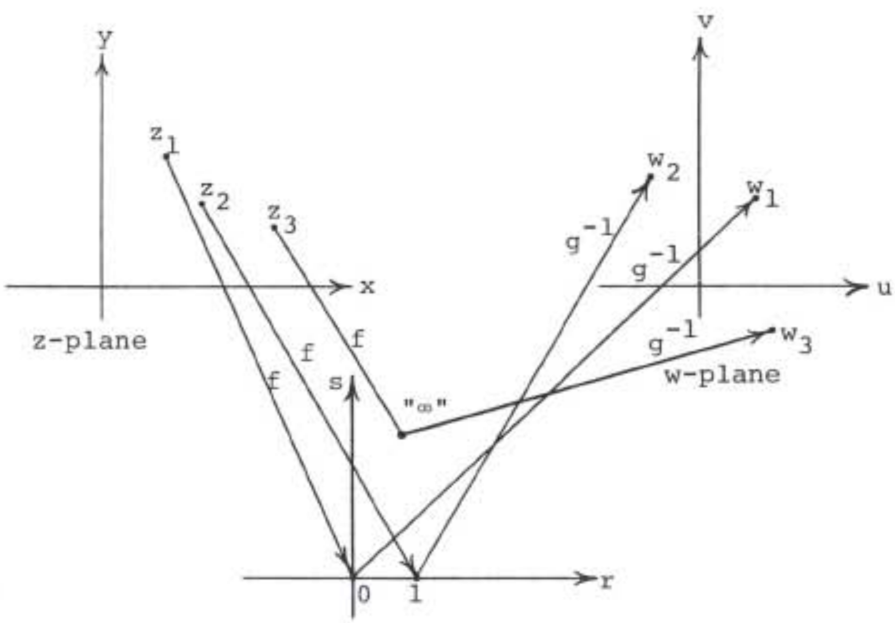
From equation (9), we see that $f \circ g^{-1}$ maps 0 into 0, 1 into 1, and ∞ into ∞. Hence, by part (b), $f \circ g^{-1}$ is the identity mapping. In other words, $g^{-1} = f^{-1}$ or, equivalently,

$$f = g \tag{10}$$

and since g was any member of the linear group for which $g(z_1) = 0$, $g(z_2) = 1$, and $g(z_3) = \infty$, equation (10) shows that g must equal f; so f is unique.

d. With part (c) as our basic building block, we now establish the more general result asked for here.

Intuitively, our technique is simply to use part (c) to construct the function f which maps z_1 into 0, z_2 into 1, and z_2 into ∞. We then find the function g which maps w_1 into 0, w_2 into 1, and w_3 into ∞. Then $g^{-1} \circ f$ maps z_1 into w_1, z_2 into w_2, and z_3 into w_3. Moreover, since f and g are unique, so also is $g^{-1} \circ f$.



Now the main consequence of this result is that z_1, z_2, and z_3 determine a circle unless the three points are all on a straight line. A similar result applies to w_1, w_2, and w_3. Hence, members of the linear group map circles or lines onto circles or lines.

1.6.11 continued

Rather than supply any further details, we conclude this exercise with a specific example. Namely,

e. We want to find h such that

$$0 \overset{h}{\rightarrow} 1$$

$$i \overset{h}{\rightarrow} -1$$

$$-i \overset{h}{\rightarrow} 0$$

Mimicking the procedure of part (d), we make the auxilliary mapping

$$0 \overset{f}{\rightarrow} 0 \overset{g}{\leftarrow} 1$$

$$i \overset{f}{\rightarrow} 1 \overset{g}{\leftarrow} -1$$

$$-i \overset{f}{\rightarrow} \infty \overset{g}{\leftarrow} 0$$

whereupon $h = g^{-1} \circ f$.

Now to find f, we proceed as in part (c). Namely, the numerator of f(z) must be 0 when z = 0. Hence, the numerator of f(z), up to a constant factor, must be z itself. Secondly, the denominator of f(z) must be 0 [i.e., $f(z) = \infty$] when z = -i. Hence, the denominator is z - (-i) = z + i. Therefore,

$$f(z) = \frac{kz}{z + i} \tag{11}$$

and since f(i) = 1, equation (11) implies that

$$1 = f(i) = \frac{ki}{i + i} = \frac{k}{2}$$

so k = 2.

1.6.11 continued

Hence,

$$f(z) = \frac{2z}{z + i}. \tag{12}$$

A similar analysis implies that

$$g(z) = \frac{k(z - 1)}{z} \tag{13}$$

and since $g(-1) = 1$, equation (13) implies

$$1 = g(-1) = \frac{k(-1 - 1)}{-1} = 2k$$

so $k = \frac{1}{2}$, and, consequently,

$$g(z) = \frac{z - 1}{2z} \tag{14}$$

We now invert g. That is,

$$w = g(z) = \frac{z - 1}{2z}$$

implies that

$$2wz = z - 1$$

or

$$(2w - 1)z = -1.$$

Hence,

$$z = \frac{-1}{2w - 1}. \tag{15}$$

From (15), we have that

$$g^{-1}(z) = \frac{-1}{2z - 1}. \tag{16}$$

1.6.11 continued

From (12) and (16), we conclude that

$$\left.\begin{aligned} g^{-1}\circ f &= \begin{pmatrix} 0 & -1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 1 & i \end{pmatrix} \\ &= \begin{pmatrix} -1 & -i \\ 3 & -i \end{pmatrix} \end{aligned}\right\} \qquad (17)$$

or, letting $h = g^{-1}\circ f$ our coding device in (17) yields

$$h(z) = \frac{-z - i}{3z - i}. \qquad (18)$$

To check (18), observe that

$$h(0) = \frac{-0 - i}{3(0) - i} = \frac{-i}{-i} = 1$$

$$h(i) = \frac{-i - i}{3i - i} = \frac{-2i}{2i} = -1$$

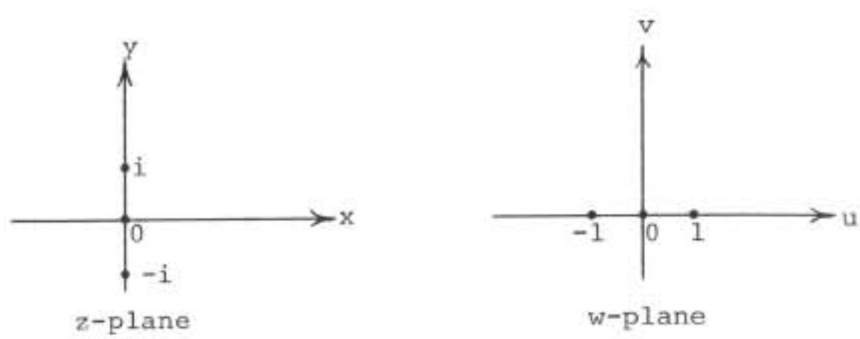
$$h(-i) = \frac{-(-i) - i}{3(-i) - i} = \frac{0}{-4i} = 0.$$

What this tells us is that

$$h(z) = \frac{-z - i}{3z - i}$$

is the only member of the linear group that maps $0\to 1$, $i\to -1$, and $-i\to 0$.

Notice that 0, i, and -i determine the y-axis while 1, -1, and 0 determine the u-axis. That is,



1.6.11 continued

Hence,

$$h(z) = \frac{-z - i}{3z - i}$$

is the only member of the linear group which maps the imaginary axis onto the real axis such that $0 \to 1$, $i \to -1$, $-i \to 0$.

[There are, of course, infinitely many members of the linear group which map the imaginary axis onto the real axis, each determined by the images of any three points on the imaginary axis.]

f. Since one of our main purposes in this block is to relate complex variables to real situations, it is appropriate that we conclude this exercise with an application that has no direct bearing on any knowledge of the complex numbers.

In this exercise, notice that the problem is well-defined without reference to the complex numbers (although the word "conformal" probably gives us a bit of a hint to think in terms of the complex numbers). The point is that if we identify the xy-plane with the z-plane and the uv-plane with the w-plane, this exercise asks us to find the conformal linear function $f(z)$ defined by $f(0) = 1$, $f(i) = -1$, and $f(-i) = 0$. In this form, we see at once that our present exercise is just a restatement of the problem given in part (e).

In particular, then,

$$f(z) = \frac{-z - i}{3z - i}$$

and if we now let $z = x + iy$, we obtain

$$f(x + iy) = \frac{-(x + iy) - i}{3(x + iy) - i}$$

$$= \frac{-x - i(y + 1)}{3x + i(3y - 1)}$$

$$= \frac{[-x - i(y + 1)][3x - i(3y - 1)]}{[3x + i(3y - 1)][3x - i(3y - 1)]}$$

1.6.11 continued

$$= \frac{[-3x^2 - (y + 1)(3y - 1)] + i[x(3y - 1) - 3x(y + 1)]}{9x^2 + (3y - 1)^2}$$

$$= \frac{-3x^2 - 3y^2 - 2y + 1}{9x^2 + (3y - 1)^2} + i\left[\frac{-4x}{9x^2 + (3y - 1)^2}\right].$$

Hence,

$$\left.\begin{aligned} u(x,y) &= \frac{-3x^2 - 3y^2 - 2y + 1}{9x^2 + (3y - 1)^2} \\ v(x,y) &= \frac{-4x}{9x^2 + (3y - 1)^2} \end{aligned}\right\} \quad (19)$$

[Notice that when $x = 0$ and $y = \frac{1}{3}$, equation (19) is not defined. This corresponds to the fact that $f(\frac{1}{3}i) = \infty$ when $f(z) = \frac{-z - i}{3z - i}$.] To find the image of the x-axis, we let $y = 0$ in (19) and obtain

$$\left.\begin{aligned} u &= \frac{-3x^2 + 1}{9x^2 + 1} \\ v &= \frac{-4x}{9x^2 + 1} \end{aligned}\right\} \quad (20)$$

To eliminate x in (20), we observe, for example, that

$$u = \frac{-3x^2 + 1}{9x^2 + 1}$$

implies that

$$9ux^2 + u = -3x^2 + 1$$

or

$$x^2 = \frac{1 - u}{3 + 9u}. \quad (21)$$

1.6.11 continued

[Notice that since $x^2 \geqslant 0$, equation (21) yields a good deal of information about u, but we shall not utilize this here.]

We then use (21) in $v = \dfrac{-4x}{9x^2 + 1}$ to obtain

$$v = \frac{-4\sqrt{\dfrac{1-u}{3+9u}}}{\dfrac{9(1-u)}{3+9u} + 1}$$

$$= \frac{-4\sqrt{\dfrac{1-u}{3+9u}}}{\dfrac{12}{3+9u}}$$

$$= \frac{-4\sqrt{1-u}}{\sqrt{3+9u}} \times \frac{3+9u}{12}$$

$$= -\frac{1}{3}\sqrt{(1-u)(3+9u)} \qquad (22)$$

Squaring both sides of (22) yields

$$v^2 = \frac{1}{9}(1-u)(3+9u)$$

or

$$9v^2 = 3 + 6u - 9u^2$$

or

$$9u^2 - 6u - 3 + 9v^2 = 1.$$

Therefore,

$$u^2 - \frac{2}{3}u - \frac{1}{3} + v^2 = \frac{1}{9}$$

and completing the square now yields

1.6.11 continued

$$\left(u^2 - \frac{2}{3}u + \frac{1}{9}\right) - \frac{4}{9} + v^2 = \frac{1}{9}$$

or

$$\left(u - \frac{1}{3}\right)^2 + v^2 = \frac{5}{9}$$

so that the image of the x-axis is the circle of radius $\frac{1}{3}\sqrt{5}$ centered at $(\frac{1}{3},0)$.

This checks with our earlier remark that linear mappings carry lines (or circles) into lines or circles.

Note

We should be a bit careful about our new use of the word "linear." Before we emphasized the idea of conformal mappings, we agreed that any mapping of the xy-plane into the uv-plane defined by

$$\left.\begin{aligned} u &= ax + by \\ v &= cx + dy \end{aligned}\right\} \qquad (23)$$

where a, b, c, and d were all real numbers, would be called linear mappings. If we insist that a linear mapping be conformal then as we saw in Exercise 1.6.4, a, b, c, and d could no longer be chosen at random. More specifically, if (23) represents a conformal mapping then

$$u = ax + by$$

implies

$$v = -bx + ay.$$

This, in turn, tells us that once we specify that (0,0) be mapped into (0,0), we have only one more point whose image we can pick at random (rather than the additional two points which we ordinarily need if the mapping is not necessarily conformal).

1.6.11 continued

Thus, if we want the mapping to be conformal and still require three points to determine it, then we must accept the broader interpretation of linear as used in this exercise. That is, by f being linear, we mean

$$f(z) = \frac{az + b}{cz + d},$$

where a, b, c, and d are complex.