Unit 4: Complex Functions of a Complex Variable

1. Overview

The main aim of this Block is to study the calculus of functions of a complex variable. Just as was the case when we studied real variables, our approach is first to discuss the number system, and then to apply the limit concept to these functions.

In our first three units we have developed the complex number system in some detail and you should now feel a bit more at home with the concept of this number system. In this unit, as our title implies, we shall discuss functions defined on the complex numbers; and the remainder of the Block will then be devoted to the topics usually identified with calculus.

You will notice that there is no lecture for this unit. The reason is that from a geometrical point of view, as we shall show in the exercises, the study of complex functions of a complex variable is equivalent to the real problem of describing mappings of the xy-plane into the uv-plane. The only difference, in terms of the language of the Argand diagram, is that the xy-plane becomes known as the z-plane while the uv-plane becomes known as the w-plane. Then in a way analogous to the notation of writing $y = f(x)$ in the study of real functions of a real variable, we write $w = f(z)$ when we are studying a complex functions of a complex variable.

2. Skim Thomas, Section 19.3. (After the exercises you may wish to re-read this section in greater detail, but our initial aim is for you to read just enough to get a quick overview of what this unit deals with. Then you should proceed directly to the exercises since our feeling is that the best way to learn this topic is in terms of working specific exercises. If there are still certain points bothering you after you have completed this unit, you may be helped by the lecture of the following unit which begins with a review of the concepts in this unit.)

3. Exercises

1.4.1(L)

a. In terms of the Argand diagram, discuss the set S if S is the subset of complex numbers defined by

$$S = \{z : z = \cos t + i \sin t,\ 0 \le t \le \pi\}.$$

b. Let C denote the complex numbers and suppose that $f:C \to C$ is defined by $f(z) = z^2$. With S as in part (a), describe the image of S with respect to f.

1.4.2

Let S be the region of the z-plane which consists of the unit circle centered at the origin between $\theta = 0°$ and $\theta = 90°$; let T be the line of slope 1 which passes through the origin; and lies in the first quadrant, and let $f:C \to C$ be defined by $f(z) = z^3$.

a. Use polar coordinates to find the image of S and T with respect to f.

b. Write f(z) in the form $u(x,y) + i\,v(x,y)$ where u and v are real-valued functions of the real variables x and y.

1.4.3

Find the real and imaginary parts of w as functions of x and y if $w = f(z)$ and

a. $f(z) = 2z$
b. $f(z) = \bar{z}$
c. $f(z) = |z|$
d. $f(z) = z^2 + 2z + i$
e. $f(z) = \frac{1}{z}$ $(z \neq 0)$

1.4.4(L)

Interpret each of the following geometrically.

a. $f(z) = \bar{z}$

b. $f(z) = z + C$ where C is a given complex number

c. $f(z) = Cz$ where C is a given complex number.

1.4.5

a. Describe, geometrically, the image of f if $f: C \to C$ is defined by $f(z) = \left(\frac{1 + i}{\sqrt{2}}\right) z + i$.

b. Describe f in the form

$$\begin{cases} u = u(x,y) \\ v = v(x,y) \end{cases}$$

1.4.6

S is the region of the z-plane consisting of the portion of the circle of radius 2 centered at the origin between $\theta = 0°$ and $\theta = 60°$. Describe the curve in the w-plane defined by $w = z^4 + 3 + 4i$ where $z \varepsilon S$.

1.4.7

Suppose $f(z) = z^3$, compute $\lim\limits_{z \to (1 + i)} f(z)$

1.4.8

Compute

$$\lim_{h \to 0} \left[\frac{(z + h)^2 - z^2}{h}\right]$$

where h is a complex variable.

The last two exercises are optional, but we hope that you will study them the same as if they had been required problems. The reason they are optional is that they have no direct bearing on the study of complex-valued functions of a complex variable. They are important, however, to explain why we do not spend too much time studying complex valued functions of a real variable or real valued functions of a complex variable. In other words, once the real numbers R and the complex numbers C each exist in their own right, there are four rather natural types of functions to study. Namely: (1) $f:R \to R$, (2) $f:R \to C$, (3) $f:C \to R$, and (4) $f:C \to C$. Case (1) was discussed in Part 1 of our course and Case (4) is the discussion of this Block. The optional exercises try to explain why we omit a special treatment of cases (2) and (3).

1.4.9

a. Let t denote a real variable. Compute $f'(t)$ if $f(t) = t + t^2 i$.

b. Find $f(t)$ if $f'(t) = t^2 + e^{3t} i$ and $f(0) = 1 + i$.

1.4.10

Suppose $y = f(z)$ [i.e., f is a real-valued function of a complex variable]. Show that if $f'(z)$ exists then $f'(z)$ <u>must be identically equal to zero</u>.

Unit 5: Differentiating Complex-Valued Functions

1. Overview

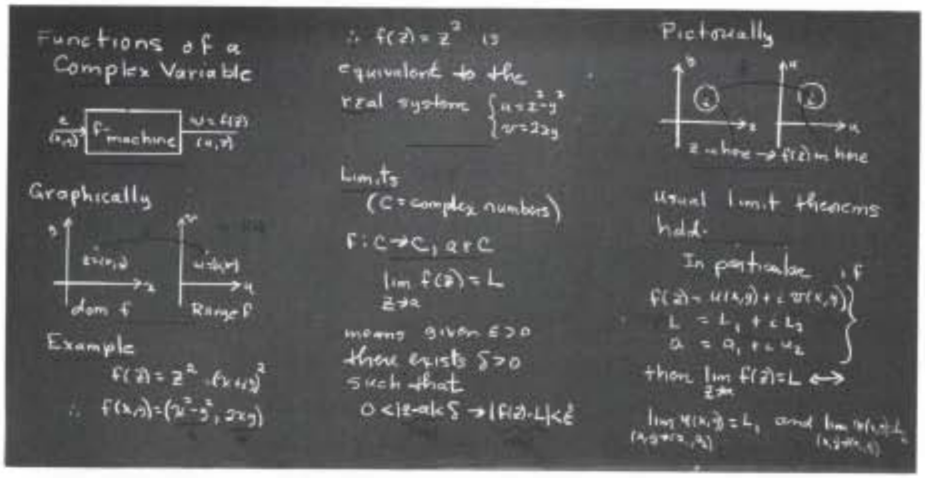
In this unit, we investigate the impact of the definition

$$f'(z) = \lim_{h \to 0} \left[\frac{f(z + h) - f(z)}{h}\right] \tag{1}$$

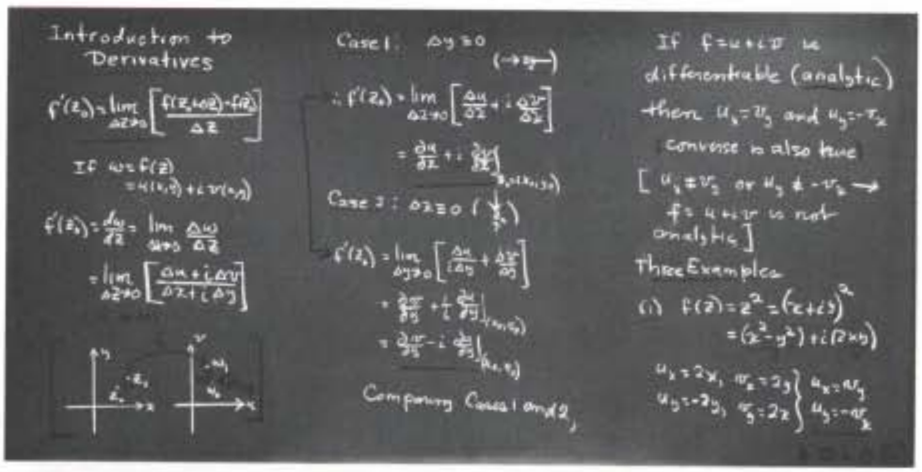
obtained by taking the definition of $f'(x)$ and everywhere replacing x by z. Notice that since h is a complex number, the quotient described in (1) requires our new complex-oriented definition of vector products, since until this time we have not been able to talk about the quotient of two vectors. [Notice that if we give equation (1) a vector interpretation, we can talk about the derivative of a vector function of a vector variable.]

At any rate, what we do in this unit is explain equation (1) from both an arithmetic and a geometric point of view and indicate how this definition has practical application in the real world.

2. Lecture 1.020



a.



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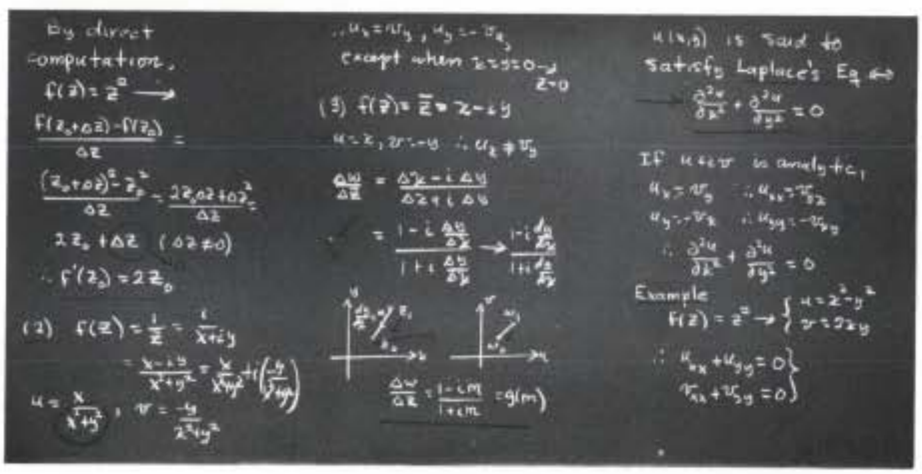

c.

3. Read Thomas, Sections 19.4 and 19.5.

4. Exercises:

1.5.1(L)

Let $f(z) = z^3$.

a. Determine $f'(z)$.

b. Write $f(z)$ in the form $u + iv$ and show that u and v satisfy the Cauchy-Riemann conditions.

c. Write $f'(z)$ in the form $r + is$ where r and s are of the form u_x, u_y, v_x, or v_y.

1.5.2

Let $f(z) = z^3 + z^2 + z + 1$.

a. Compute $f'(z)$.

b. Write $f(z)$ in the form $u + iv$ and show that u and v satisfy the Cauchy-Riemann conditions.

c. Use part (b) to compute $f'(z)$ and check your answer with the result obtained in part (a).

1.5.3 (This is a replica of the previous exercise except that we have chosen f so as to complicate the arithmetic a bit more.)

Suppose f is defined on the set of all complex numbers, excluding 0 (i.e., dom $f = c - \{0\}$) by $f(z) = \frac{1}{z^2}$.

a. Compute $f'(z)$.

b. Write $f(z)$ in the form $u + iv$ and show that u and v satisfy the Cauchy-Riemann conditions.

c. Use part (b) to compute $f'(z)$ and show that this answer is the same as that found in part (a).

1.5.4(L)

Suppose that $u(x,y) + iv(x,y)$ denotes an analytic function. Use the Cauchy-Riemann conditions to conclude that both u and v satisfy Laplace's equation; i.e., $u_{xx} + u_{yy} = 0$ and $v_{xx} + v_{yy} = 0$.

1.5.5(L)

Let $u = x^4 - 6x^2y^2 + y^4$.

a. Show that u satisfies Laplace's equation.

b. Find v(x,y) such that u + iv is analytic.

c. Find v(x,y) such that v + iu is analytic.

1.5.6

Let $u = x^3y^4$.

a. Use Laplace's equation to show that u can never be the real (or imaginary) part of an analytic function.

b. Verify the result of part (a) by showing that with u as above, there is no function v = v(x,y) which can satisfy the Cauchy-Riemann conditions.

1.5.7(L)

Suppose u(x,y) + iv(x,y) is analytic. Prove that the family curves u(x,y) = constant intersect the members of the family v(x,y) = constant at right angles except possibly when $(u + iv)^1 = 0$.

1.5.8

Use the fact that $\frac{d(z^2)}{dz} = 2z$ to conclude that the family of curves $x^2 - y^2 = c_1$ and the family of curves $2xy = c_2$ always intersect at right angles except when $c_1 = c_2 = 0$.

1.5.9 (Optional)

[This exercise is optional in the sense that if you are willing to take the result for granted then there is no loss in continuity of the subject if you omit this proof. On the other hand, the result is of sufficient importance that the interested student should try to follow the proof. Apart from any other considerations, the proof reviews some important aspects about derivatives of real functions of two real variables. In fact, the study of complex functions of a complex variable would have justified the calculus of real valued functions of two real variables, had there not already have been such justification.]

Suppose that u and v are continuously differentiable functions of x and y; and that u and v satisfy the Cauchy-Riemann conditions. Prove that the function defined by u + iv is analytic.

[This exercise is the converse of what was proved in the lecture and in the text. There we showed that if u + iv was analytic, then u and v satisfied the Cauchy-Riemann conditions.]

1.5.10 (Optional)

[This exercise is optional in the sense that the material covered by it is the subject of the entire next unit. We include it here not only as a forerunner of the next unit, but because we feel it is as important to get a feeling for differentials and the geometric interpretation of a derivative in the complex case as it was in the real case.]

a. If $w = f(z)$ is analytic at $z = z_o$, show that in any neighborhood of z_o,

$$\Delta w = \left.\frac{dw}{dz}\right]_{z=z_o} \Delta z + k\Delta z, \quad \lim_{\Delta z \to 0} k = 0.$$

b. Assuming that the approximation

$$\Delta w = \left.\frac{dw}{dz}\right]_{z=z_o} \Delta z$$

is valid, interpret what it means for f to be analytic, in terms of how it maps the z-plane into the w-plane.

c. Describe the image of a sufficiently small neighborhood of the point (1,1) under the mapping

1.5.10 continued

$$\begin{cases} u = x^2 - y^2 \\ v = 2xy \end{cases}$$

[Note how part (c) makes no direct reference to complex numbers.]