Unit 6: The Chain Rule, Part 2

1. (Optional) Read Thomas, Section 15.12. (See Preface to Exercise 3.6.2 for an explanation.)

2. Exercises:

3.6.1(L)

Suppose $w = f(x,y)$ where f is a continuously differentiable function of x and y. Assume that w_{xx}, w_{yx}, w_{xy}, and w_{yy} all exist.

a. Show that if $x = r \cos\theta$ and $y = r \sin\theta$ then

$$\frac{\partial^2 w}{\partial r^2} = \frac{\partial^2 w}{\partial x^2}\cos^2\theta + \frac{\partial^2 w}{\partial y^2}\sin^2\theta + \left(\frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 w}{\partial y \partial x}\right)\sin\theta\cos\theta.$$

b. In part (a), let $f(x,y) = x^2y^3$. Compute $\frac{\partial^2 w}{\partial r^2}$ first by expressing x^2y^3 in polar coordinates and then by using the result expressed in part (a).

3.6.2 (Optional)

[In this exercise, we intend to show specifically a function f defined in the entire xy-plane which has the property that f, f_x, and f_y exist and are continuous in a neighborhood of (0,0). Moreover, f_{xy} and f_{yx} also exist at (0,0) but $f_{xy}(0,0) \neq f_{yx}(0,0)$. The existence of such a function f is sufficient to warn us to beware about assuming that f_{xy} and f_{yx} are the same.

The reader who is willing to accept this precaution loses nothing but valuable experience in working with partial derivatives by omitting this exercise. On the other hand, if you can follow the presentation, it should strengthen your concept of how one proceeds to take higher order partial derivatives.

Before we get to the stage of not seeing the forest because of the trees, let us hasten to point out the major result that for most functions f, if f_{xy} and f_{yx} exist at the point (a,b) then

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3.6.2 continued

$f_{xy}(a,b) = f_{yx}(a,b)$. In particular, the theorem governing this result tells us that if f, f_x, f_y, and f_{xy} exist <u>and are continuous</u> in a neighborhood of the point (a,b), then f_{yx} also exists at (a,b) and in fact $f_{yx}(a,b) = f_{xy}(a,b)$. We omit the proof of this theorem, but again, the interested reader may study the proof if he so desires since it is presented in Section 15.12 of the Thomas text.

Our major concern, regardless of whether you try this exercise, is to make certain you appreciate the need for the existence and continuity of f_x, f_y, f_{xy}, etc.

<u>Note</u>: The expression $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$ is sometimes abbreviated by $\frac{\partial^2 f}{\partial y \partial x}$ rather than by $\frac{\partial^2 f}{\partial x \partial y}$. Which interpretation is used is not important, but consistency is. That is, $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$ and $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$ need not be equal.]

Define f in the xy-plane by

$$f(x,y) = \begin{cases} \dfrac{xy(x^2 - y^2)}{(x^2 + y^2)} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Show that f, f_x, and f_y each exists and is continuous at (0,0). Then, show that f_{xy} and f_{yx} each exist at (0,0) but that $f_{xy}(0,0) \neq f_{yx}(0,0)$. Finally, show that f_{xy} is not continuous at (0,0).

3.6.3

Using the same notation as in Exercise 3.6.1(L), assume that w, w_x, w_y, and w_{xy} exist and are continuous.

a. Show that

$$w_{\theta\theta} = w_{xx}r^2 \sin^2\theta - 2w_{xy}r^2 \sin\theta\cos\theta + w_{yy}r^2\cos^2\theta - w_x r\cos\theta - w_y r\sin\theta.$$

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3.6.3 continued

b. Combine part (a) of this exercise with part (a) of Exercise 3.6.1(L) to show that

$$r^2 w_{rr} + w_{\theta\theta} = w_{xx}r^2 + w_{yy}r^2 - r(w_x \cos\theta + w_y \sin\theta).$$

c. Observing that $\cos\theta = x_r$ and $\sin\theta = y_r$, show from part (b) that if $r \neq 0$ then

$$w_{xx} + w_{yy} = w_{rr} + \frac{1}{r^2} w_{\theta\theta} + \frac{1}{r} w_r.$$

3.6.4(L)

Given that $w = f(u,v)$ where f is a continuously differentiable function of the independent variables u and v, assume that u and v are also continuously differentiable functions of x and y. Assume that w_{xx}, w_{yy}, w_{uu}, w_{uv}, w_{vv}, u_{xx}, u_{yy}, v_{xx}, v_{yy}, etc. all exist.

a. Show that

$$w_{xx} + w_{yy} = w_{uu}(u_x{}^2 + u_y{}^2) + w_{vv}(v_x{}^2 + v_y{}^2) + 2w_{uv}(u_x v_x + u_y v_y)$$
$$+ w_u(u_{xx} + u_{yy}) + w_v(v_{xx} + v_{yy}).$$

b. Use the result of (a) in the special case of polar coordinates where $u = r$ and $v = \theta$ to obtain another proof that

$$w_{xx} + w_{yy} = w_{rr} + \frac{1}{r^2} w_{\theta\theta} + \frac{1}{r} w_r.$$

c. Compute $w_{xx} + w_{yy}$ if $w = e^{x^2+y^2}\cos(x^2 - y^2)$.

d. Do part (c) using the substitution (change of variables) $u = x^2 + y^2$ and $v = x^2 - y^2$.

3.6.5(L)

a. Suppose w depends on r but not θ, say $w = h(r)$, and that h is a differentiable function of r. Show that in this case $w_{xx} + w_{yy} = h''(r) + \frac{1}{r} h'(r)$.

b. If $w = h(r)$ where $r = +\sqrt{x^2 + y^2}$, express w in terms of x and y if $w_{xx} + w_{yy} \equiv 0$.

3.6.6(L) [Actually, part (a) is not a learning exercise.]

Suppose $w = w(x,y)$ where $x = e^u \cos v$ and $y = e^u \sin v$ and w, w_x, w_y, w_{xx}, w_{yy}, and w_{xy} all exist and are continuous.

a. Show that $w_{uv} = xy(w_{yy} - w_{xx}) + (x^2 - y^2)w_{xy} - y\, w_x + x\, w_y$.

b. Use the result of (a) to solve the partial differential equation

$$xy(w_{yy} - w_{xx}) + (x^2 - y^2)w_{xy} - y\, w_x + x\, w_y = 0.$$

c. Show, in particular, why $w = \ln(x^2 + y^2) + \frac{y}{x}$ is a solution of the differential equation in part (b).

3.6.7

With w as in Exercise 3.6.6(L), let $x = u^2 - v^2$ and $y = 2uv$.

a. Show that $w_{uv} = 2(y\, w_{yy} + 2x\, w_{xy} - y\, w_{xx} + w_y)$.

b. Solve the partial differential equation

$$y\, w_{yy} + 2x\, w_{xy} - y\, w_{xx} + w_y = 0.$$