PROFESSOR: Hi. In our lesson today, we shall try to find a relationship between line integrals and double integrals. In particular, we shall equate line integrals along certain types of closed curves to double integrals over the region enclosed by the particular closed curve. In particular, the title of today's lecture is Green's theorem. And in terms of some particulars, I guess the intuitive notion that we'll have to accept is the idea of what one means by a connected region.

I think this is one of those times in mathematics where we can say if you have an intuitive idea of what the concept means, you have the exact idea of what the concept means. Essentially, connected means that you can get from any place in the region to any other place in the region without ever having to leave the region. It's as simple as that. OK. Connected means that you can get from any point in the region to any other point in the region without ever having to go outside the region.

The mathematician distinguishes between two different types of connected regions. One of which is called simply connected. The other of which being the opposite of simply connected, meaning non-simply connected or multiply connected. I prefer multiply connected to non-simply connected, because in non-simply connected you don't know whether you're modifying simply or connected. A non-simply connected region, when you say multiply connected, means that in particular the region is connected.

The intuitive way of looking at this is that simply connected means no holes. For example, this region here is not only connected, but it's simply connected. There are no holes. This region is connected. You see I can get from any point to any other point without ever having to leave the region. But it's called multiply connected, because there are holes in the region.

By the way, you see, the trouble with using expressions like holes versus non-holes, is that in that particular case, where we're dealing with two or three dimensions there's certainly no problem, because we intuitively visualize what a hole is. In n-dimensional space, and we have analogs of these results in these n-dimensional space, the problem is that how do you describe a hole without referring to a picture? And the more rigorous mathematical definition of a simply connected region is the following, the connected region, and it has to be connected before you talk about simply connected, you see, the connected region r is called simply connected if its complement is also connected.

Remember the complement is the portion of space left when the particular region is deleted. For example, here we have a simply connected region. Notice that the complement of this region would be, for example, the entire blackboard with this piece missing. And the entire blackboard with this piece missing is still a connected region. On the other hand, if we were to take this connected region and look at its complement, its complement is this little hole together with what's outside here. And notice that there is no way from getting from here to here without having to go outside the complement.

You see, in other words, the complement of the multiply connected region is not connected. At any rate, whichever way you want to look at this, is fine. Think of simply connected, at least in the two dimensional case, as meaning no holes.

And with this as background, Green's theorem, call it the theorem, I guess, is stated as follows. Suppose r is simply connected with a boundary c. Notice r is the region. c is its boundary. And a very interesting thing takes place. If you form the line integral around the boundary c, and by the way, with this loop in here, this is the symbol that we mean around the closed curve.

The arrow head is simply done to invoke the convention. And we do have to pick a convention just like we did in calculus of a single variable about positive and negative and how you move. We pick the positive direction as being the direction in which one goes around the boundary so that the region appears to our left as we go around the boundaries. As we're going around here, you see, in this direction, the region appears to our left. At any rate, I'll mention that in more detail later.

The directed line integral around closed curve c, mdx plus ndy turns out to be a double integral over the region r enclosed by the curve. And that double integral has as its integrand the partial of n with respect to x minus the partial of m with respect to y. And I will refer to this again very, very shortly. But I hope that this rings a familiar bell in terms of what we talked about exact differentials.

Remember a differential mdx plus ndy was exact if the partial of n with respect to x equaled the partial of m with respect to y. I'll return to that in a moment. For the time being, all I want you to see is what Green's theorem says.

What Green's theorem does is it relates a line integral along the boundary of a simply connected region to a double integral over the region itself. The proof of the theorem is in the Thomas text. It's available in a pretty straightforward form for anybody who would like to see the proof of the result.

I don't think I want to prove the result in this lecture. I think I would rather hammer home what the theorem says rather than what this rigorous proof is. The idea is though that what? A line integral is related to a multiple integral. And all that's required for Green's theorem to be true is that m, n, the partial of m with respect to y, the partial of n with respect to x must exist and be continuous in a region containing r.

And we'll see later that even the condition of simply connected can be waived. In fact, in the Thomas demonstration, we don't even deal with the most general simply connected regions. He gives his proof for a very specific type of simply connected region. But again, that isn't crucial either. We'll talk about that in more detail later in the lecture as well as in the exercises.

The idea again though is what? We're not relating arbitrary line integrals to arbitrary multiple integrals. But Green's theorem relates a line integral along the boundary of a simply connected region to a double integral defined on the interior of the closed curve c, the region r. And there are two interesting cases. One of which-- well, there are more than two, but two that we'd like to talk about.

One of the interesting cases we've already hinted at and that is if mdx plus ndy is exact, then the integral of mdx plus ndy around the closed curve c is 0. And the reason for this, you see, is that the integral around the closed curve, by Green's theorem, is just a double integral partial n with respect to x minus partial m with respect to y. And if mdx plus ndy is exact, then these two terms are equal so that their difference is identically 0. And to integrate 0da over any region r, of course, is just 0 itself.

In other words, then if mdx plus ndy is exact, the integral around any closed curve is 0. And by the way, this ties in with a question that was raised in our previous lecture. And that was we saw that if we had a force for example, and we had a particle move from the point p0 to the point p1, under the influence of that force that the work done as that particle moved in general dependent on the curve which connected p0 to p1.

My claim is that if the force, mi plus mj has as its components, see m and n where the partial of m with respect to y equals the partial of n with respect x. My claim is under those conditions the work is independent of the path. And that's what we mean in turn by a conservative force field.

The proof is quite simple. What we do is we pick two different paths that connect p0 to p1. We call one of the paths c1 and the other path c2. And now what we're saying is if we think of c as being the curve that keeps this region to our left as we go along here, then the integral around c is 0. But what is the integral around c? It's the integral along c1 minus the integral along c2.

And I hope that that's clear from the exercises of last time. That when you switch the sense of the path, all you do is change the sign of the integrand. In fact, by the way, while we're on that topic, let's digress for a moment and return to our statement of Green's theorem and observe that if we were to switch the convention, if we want to switch the convention and keep the positive direction as that which kept the area on our right, notice that this would change the sense here, which should give me a minus sign.

But if I interchange, you see, the order of the terms here, if I reverse the sense, the sine of the integral, the double integral, will also change. The important point being that all of these sine conventions are preserved by things like Green's theorem and the like. But the important point is, getting back to this now, the integral around the closed curve c is the integral along c1 minus the integral along c2.

And since the integral along that closed curve is 0, it says that this minus this is 0. Consequently, these two integrals are equal. And since c1 and c2 are arbitrary paths that connect p0 and p1, that shows that the line integral in this particular case does not depend on the path but only the points p0 and p1. Again, we will emphasize that more in the exercises.

The second interesting application of Green's theorem is a form which not only shows us how a line integral and a multiple integral are related, but it actually shows us a way of computing an area in terms of a line integral. In particular, if in Green's theorem we let n equal x and m equal negative y, notice that Green's theorem then becomes what?

The integral along a closed curve c, mdx plus ndy is this minus ydx plus xdy. The partial of n with respect to x is 1. The partial of m with respect to y is minus 1. Therefore the partial with n with respect to x minus the partial of m with-- the partial of n with respect to x minus the partial of m with respect to y is 1 minus minus 1, which is 2.

In other words, Green's theorem says that the integral around the closed curve in this case is twice the area of the region r. You see without the 2 in here this would just be the area of the region r. Since 2 is a constant factor, we can take it outside the integral sine.

In other words, in terms of a picture if I want to find the area of the region r and I want to use a line integral, I need only take one half of this integral. In other words, the area of the region r is one half the integral around the boundary of r, xdy minus ydx. And again, I will drill on that in the homework. The one remaining thing I wanted to point out before we go to a specific example is how we remove the restriction of the region being simply connected.

You see what I claim is that we may replace simply connected by connected. Since multiply connected regions are unions of simply connected ones with cuts, whatever that means. Let me show you what I'm saying here. Look at this multiply connected region here. What I'm saying is suppose I were to cut this region, I can then visualize this as the union of two simply connected regions.

In fact, if I have a piece of paper in my pocket, I think we might as well-- the entire two parts of calculus are almost over, and I've never done an experiment for you. Here's one I think even I can handle.

Here's a connected region. It's a piece of paper. I'll fold it. And I'll now make it. It's simply connected, because there are no holes in it. I will now undo that. I will make the region connected but now multiply connected. You see there's a hole in it.

And all I'm saying is if I cut this region, not necessarily along a straight line. All I've got to do is to cut it. If I cut this region, then what I have are two simply connected regions that can be viewed as being fit together this way. But to see them clearly, we can just separate them. But the reason I wanted to cut them for you is to see that I can actually fit these together.

And notice that the cut that joins these two from a theoretical point of view has no thickness. Right? That's just a line that joins these two. And here's the key point.

You see physically, certainly if I put a cut inside the region, the resulting picture is different than with the cut missing. In other words, I can distinguish between the shape that I've drawn without the accentuated chalk mark in here and the shape that I have with the accentuated chalk mark in here.

But here's the key point. First of all, as far as computing a double area is concerned, the cut having no thickness has zero area. So if somehow I were to compute the area of the region r without the cut, I would expect to get the same area. Or if I were to compute whatever you want to call this, the density or what have you, if I want to compute the mass of the region r with the cut, it should be the same as the mass without the cut, because the difference is essentially what? A set of points which has zero area.

And by the way, for those of you who may read ahead in mathematics and have heard such expressions as Lebesgue integrals, the Lebesgue integral and the like. These are things defined on sets of measure 0. What these things mean from an intuitive point of view is essentially what we're talking about now.

You see here are infinitely many points on this cut. And yet the important thing is that with respect to area these points, even though they're infinite in number, contribute no area. In other words, there are infinitely many points on a line, but a line having no thickness has no area. So certainly putting the cut in does not affect the double integral.

On the other hand, since we always assumed that the region is kept on our left as we go around, it turns out that from the single integral, the line integral point of view, each cut cancels because of opposite sense.

What I'm saying is if I take this region and slice it, you see, how do I traverse this piece? Since I want the area to be enclosed on my left, I traverse this region in this particular direction. When it comes to this piece, I traverse this region in this direction so that the area is enclosed to my left.

When I now superimpose these two, what happens is, since this line integral and this line integral go exactly over the same curve, remember I just separated these so I could see them better, when I push these together, this is the same line. Since they have the same function, the same points, but only opposite sense, these two integrals cancel.

In other words, if I'm given a region that's multiply connected, I can view it like this, meaning I go around this piece this way so that my region is on my left. I go around the outside border this way so that the region is on my left. And when I want to use Green's theorem, all I do is essentially, I don't even have to make the two cuts here, I could even make one cut. And look at this as the union of--

See, well, the two is-- see if I make just the one cut and leave the line out, there are no longer any holes in the region here. But that's not the important point. The important point is that to visualize the integral that I want, or the regions that I want, all I have to do is put the cut in. And whatever the cut does is canceled out in the statement of Green's theorem. In other words, again, that quite often in using Green's theorem we still apply it to multiply connected regions.

The reason that we stress simply connected is that the proofs that are given for Green's theorem are easiest to handle in the case of simply connected regions. And then we simply generalize it to multiply connected by pointing out what we just did here, that we can always visualize a multiply connected region as a union of simply connected regions if the appropriate cuts are made. And again, that will be emphasized in the exercises.

I thought that what we should do to finish off today's lesson would be to actually do a trivial exercise. By trivial I mean Green's theorem has profound application. It's more than just relating line integrals to multiple integrals. The applications of these things are enormous. And we're going to hit a few of these in the exercises.

But what I wanted to do was to pick sort of a straightforward example just to have you see what Green's theorem means when we translate into a concrete situation. And the example I have in mind is simply this. Let's compute the line integral around the closed curve c, y cubed dx plus x to the fourth dy, where the curve c happens to be the following simply closed, our closed curve.

I start at the origin and move to the point 1,1 along the parabola y equals x squared. Then I moved from 1,1 to 0,1 along the line y equals 1. And then I move from 0,1 down to 0,0 along the line x equals 0. The boundary of this is what I'm calling my region c, my curve c. Notice that the region r enclosed by c lies on my left as I traverse the boundary of c. And what I want to do is to find the value of this line integral along this particular closed curve c.

And again, if you want to see this in terms of our lecture of last time, where we talked about work and force, imagine that my force is defined in the xy plane at any point x,y to be y cubed i plus x to the fourth j, et cetera. Meaning dr, as usual, will be dxi plus dyj. And what we're saying is that this particular integral represents the work that would be done if a particle were starting at the point 0,0 and being moved along the curve c so that it returns to the point 0,0.

Now keep in mind, merely because the particle is returning to the same point from which it started is not enough to guarantee that the work done is 0. In fact, the guarantee that would be necessary would be that the partial of y cubed with respect to y would have to be identical to the partial of x to the fourth with respect x. And that certainly isn't the case in this particular example. We don't have a conservative force field in this case.

In this case, we suspect that the work done should depend on the path connecting the points. In particular, if we go around a closed path, the work done would not have to be 0. At any rate, that's enough lip service paid to the physical interpretation of this. Let's actually try to solve this problem.

And what we're going to do is solve it in two ways. The first way will be to emphasize Green's theorem. We'll solve it first by Green's theorem, because that is the sermon for today. And the second method will be to show how we would have solved the same problem if we didn't have Green's theorem. The idea being I won't tell you which of the two ways one should have done this. The important point from our point of view is that the two methods yield the same answer.

At any rate, to use Green's theorem notice that m is y cubed. n is equal to x to the fourth. So the statement of Green's theorem, which says that the integral around the closed curve c, mdx plus ndy is the double integral around the region enclosed by c. The partial of n with respect to x minus the partial of m with respect to yd, that leads to this.

And now, what is my region a? I'll think of it as being a dydx. And if I look at my region a, I observe that for a fixed value of x, y varies from the curve x squared to 1. And x, in turn, varies any place from 0 to 1. So the line integral that I'm trying to evaluate is this particular double integral. By the way, we don't really need this, what comes next, because from here on in, this should be old hat.

But just by way of review, notice that I could have written this with the order of integration reversed so I wouldn't have to worry about an x squared in the lower limit. I might like all of my lower limits to be zero. Notice I could have picked a horizontal strip and shown that for a fixed value of y, x varies from 0 to the square root of y. And y varies from 0 to 1.

In any event, this is just garnishing. This is the easy part. All I'm saying is we now evaluate this iterated integral, integrating with respect to x. The integrand becomes x to the fourth minus 3xy squared evaluated between x equals 0, and x equals the square root of y. That, of course, leads to y squared minus 3y to the 5/2.

I integrate that between 0 and 1. I get 1/3 y cubed minus 6/7y to the 7/2. And as y goes from 0 to 1, this becomes minus 11/21. By the way, again, I hope it's clear from previous exercises that the minus sign simply has physical significance. That all we're saying is, is that one would expect that if the motion takes place in the direction of the force, the force is helping us move the particle. Whereas if the force is taking place opposite to the direction of motion, we have to push against the force to get the particle there.

So what we're saying is in one case, the sense is positive. The other is negative. I'm not going to worry here physically about what the meaning of the minus sign is. But using Green's theorem, we have shown what? That the integral around the closed curve c, as we've defined c before, y cubed dx plus x fourth dy, is minus 11/21.

And what I would like to do now to finish today's lesson is to do the same problem. Only doing it without Green's theorem. And again, the point is to do this problem without using Green's theorem. Remember what my region was over here. Without having to refer back to the previous work. What I'm doing is my curve c consists of three separate line integrals.

c1 is the curve y equals x squared as x goes from 0 to 1. c2 is the curve y equals 1, where x varies from 1 to 0. And the curve c3 is x equals 0, and y varying continuously from 1 to 0.

At any rate, using the fact that if a curve is the union of curves that the line integral around that curve is the sum of the integral over the curves making up the union. I get that the line integral along c, y cubed dx plus x to the fourth dy, is the sum of these three line integrals. In other words, I integrate along c1, c2, and c3, where c1 is y equals x squared. And x varies continuously from 0 to 1.

c2 is the line y equals 1. And x varies continuously from 1 to 0. c3 is the line x equals 0. And y varies continuously from 1 to 0, using the concepts that we talked about in the last lecture. For example, to evaluate this integral this will be what? I replace y by x squared, dy by 2xdx and integrate as x goes from 0 to 1. So my first integral is simply integral from 0 to 1, x to sixth dx plus x to the fourth times 2xdx.

My second integral coming back to here, y is constantly 1. All right. So dy is 0. That makes this term drop out. Consequently, this term becomes what? 1dx as x varies continuously from 1 to 0. So that integral becomes integral 1 to 0 dx. And my final integral, over c3, notice for that integral x is 0. So both this term and this term drop out.

See x is 0. So this term drops out. The fact that x is a constant makes dx equal to 0. So this term drops out. So my integrand is just 0dy as y goes from 1 to 0. And to make a long story short here, this integrand is simply x to the sixth plus 2x to the fifth dx from 0 to 1.

This is simply another way of writing minus 1. This is 0. So my line integral along the closed curve c is 1/7x to the seventh plus 1/3x to the sixth evaluated between 0 and 1 minus 1. This comes out to be a 1/7 plus 1/3, which is 10/21 minus 1 makes this minus 11/21. And hopefully, that checks with the answer that we got by the previous method.

And at any rate, if nothing else, that is an application as to how Green's theorem works. Now what this does as far as this lecture is concerned, and the exercises will bring this out, is it ties in line integrals with multiple integrals. That will be the last lecture of this particular block of material.

And in the next lesson, what we will do is introduce a new block of material. We in particular shall talk about complex numbers. And that will occupy us for a little while. And at any rate though until next time, goodbye.

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