PROFESSOR: Hi. Today, we're going to generalize our discussion of series to cover series of functions rather than series of just constants. In other words, you'll notice that, up until now, what we've been doing in our discussion of series was have a bunch of fixed numbers and then add them. You see, in the same way that one starts with constants and ordinary arithmetic and then starts talking about functions, the same thing occurs here, that maybe we are now interested in a series over a range of different values of 'x'.

Now, rather than to talk abstractly about this, I'd like to return to a topic that we touched upon briefly much earlier in the course, and then come back to that topic with far greater power than we were able to exhibit up until now. And it's the idea of approximating a function by polynomials. In other words, this question of degree of contact. And again, rather than review that abstractly, let's talk in terms of a picture.

You see, today's lecture is called polynomial approximations, and the idea is this. Let's suppose we have the curve, 'y' equals 'f of x'. It's smooth. It passes through the origin. And we're told the following. We say, look, we're interested in approximating this curve in a neighborhood of 'x' equals 0-- in other words, at this point here-- by various polynomials.

Now, the simplest polynomial, of course, is a constant. In other words, the question is, how should we choose a straight line, 'y' equals a constant, 'y' equals ''P sub 0' of x'-- in other words, a polynomial of degree 0, a constant-- if we want the best possible horizontal line that approximates the curve at this particular point? And I think you can see that it's rather trivial at this case to say, OK, let's take the horizontal line that passes through this point. See, any other horizontal line misses this point altogether.

OK, so far, so good. Then somebody says, now, look it, if we remove the restriction that the line be horizontal, what about any straight line? What straight line has the best degree of contact with the curve at this particular point? In other words, what first degree polynomial has the highest degree of contact over here?

Now again, sparing you the details-- because the details are supplied in the text-- the polynomial there is simply what? 'f of 0' plus ''f prime of 0' times 'x''. In other words, it's a tangent line. The basic equation of a tangent line is 'y' equals 'mx plus b', where 'm' is the slope, which is 'f prime of 0', and 'b' is the y-intercept, which is 'f of 0' over here.

Now, the next question comes up. What if we wanted a quadratic polynomial, a second degree polynomial that fits the curve even better, in other words, has a greater degree of contact at this particular point?

Let's label that 'y' equals ''P sub 2' of x' to indicate a second degree polynomial. And to analyze what that polynomial must look like, let's simply write down some undetermined coefficients here, namely, let's let 'P2 of x' be 'a0 plus 'a1x' plus 'a2 'x squared''. And the idea is to determine what 'a0', 'a1', and 'a2' must look like if this polynomial is to have a maximum degree of contact with the curve, 'y' equals 'f of x'.

Now, what do you mean by maximum degree of contact? Well, what you mean is you want the function at 0 to equal the given function at 0. You want the first derivative of 'P2' evaluated at 0 to equal the derivative of 'f' evaluated at 0. And you want the second derivative of 'P2' evaluated at 0 equal to ''f double prime' of 0'. Well, let's go through this in slow motion.

First of all, 'P2 of 0' is simply 'a0'. 'P2 prime' is simply 'a1 plus '2 a2 x'', and evaluating that when 'x' equals 0, we just have 'a1'. So 'f of 0' must equal 'a0'. 'a1' must equal 'f prime of 0'.

Now, what about the second derivative? Notice that every time we differentiate this polynomial, another term drops out. 'a0' drops out the first time, 'a1' the second time. And when you've differentiated this term twice, all you have left is '2 a2'. In other words, 'P2 double prime' evaluated at 0 is '2 a2'. That must be 'f double prime' of 0. Equating these two, 'a2' must be the second derivative of 'f' evaluated at 0 over 2.

I inadvertently wrote in 2 factorial here, which is the right answer, but I haven't led up to that yet, so if you want to call this 2, that's fine. At any rate, here's what I've shown. What I've shown is that, if you want the polynomial of degree 2 that fits the curve the best, in a neighborhood of 0 comma 'f of 0', this is how the coefficients must be chosen. And in fact, this situation generalizes very nicely.

You see, suppose you want the n-th degree polynomial expression. Differentiate this 'n' times. You see, each time you differentiate, one of these terms drops out. What happens to this term as you differentiate it? The first time you differentiate it, you bring down an 'n'. The next time you differentiate, you bring down an 'n minus 1'. By the time you've differentiated 'n' times, this is 'x to the 0', that's 1, 'a sub n' is still here, and 'n' times 'n minus 1' times 'n minus 2', et cetera, is just what we call 'n factorial'. In other words, the n-th derivative of ''P sub n' of x' is 'n factorial' times 'a sub n'. If you evaluate that at 0, that's still 'n factorial' times 'a sub n'.

That must also equal the n-th derivative of the function evaluated at 0, by definition of what you mean an n-th degree of contact. At any rate, that says that the n-th coefficient, the coefficient of 'x to the n', must be the n-th derivative of 'f' evaluated at 0 over 'n factorial'. By the way, notice what this says. And we'll come back to this much more strongly later. It says that, for this coefficient to exist, the n-th derivative of 'f' at 0 has to exist. In other words, if 'f' does not possess its n-th derivative, this particular equation doesn't make sense.

And I'll show you what that means in a little while. But at any rate, notice that what we've now shown is that ''P sub n' of x'-- if you want to use the sigma notation-- can be written how? Well, it's the sum as 'k' goes from 0 to 'n', the k-th derivative of 'f' evaluated at 0 over 'k factorial', times 'x to the k-th' power. Which written out longhand is simply what? 'a0' plus 'a1x' plus ''a2 'x squared'' over '2 factorial'' plus et cetera, the n-th derivative of 'f' evaluated at 0 over 'n factorial' times 'x to the n'.

And what we do is we let 'P of x' denote the limit of ''P sub n' of x' as 'n' goes to infinity. In other words, for a fixed 'x', we look at what the sequence converges to as 'n' goes to infinity, and we denote that as a series. You see, what happens is 'x' is a variable here, but if you replace 'x' by a specific number, what's inside the summation sign becomes a constant which depends only on 'n', and we're back to our study of ordinary series again.

Now, what does 'P of x' mean in this case? Going back to our diagram, it means that, as we let n get bigger and bigger and we determine more and more coefficients, we get what? Hopefully, what's a better fit to the curve over here, that we get a greater degree of contact every time we tack on the next term in the series. Now, because this may seem a little bit abstract, let's illustrate this thing numerically.

Let's pick a specific example. And I'll pick an easy one to compute. Let 'f of x' be 'e to the x'. After all, what could be easier? Because the derivative of 'e to the x' with respect to 'x' is just 'e to the x', so the n-th derivative of 'e to the x' is always 'e to the x'. When 'x' is 0, 'e to the 0' is 1. So the n-th derivative of 'e to the x', evaluated at 0, is 1. If I divide that by 'n factorial', I just get

'1 over 'n factorial''. In other words, the sequence of polynomials that approximates 'e to the x' is given by what? Summation ''x to the k' over 'k factorial'' as 'k' goes from '1 to n'.

And again, what this means is what? The first approximation is the line, 'y' equals 1. The straight line approximation for the highest degree of contact at the origin is 'y' equals '1 plus x'. The best quadratic approximation is 'y' equals '1 plus x plus ''x squared' over 2'. The best cubic fit is when 'y' is '1 plus x plus ''x squared' over 2', plus 'x cubed over 6'', 6 being 3 factorial. And we can continue on this way, getting better fits as we add more and more terms.

And again, I've spoken quite rapidly, the reason being that this part is done superbly in the textbook, complete with graphs, tables, and what have you. Now, here's the question. The question is, look it, we can compute 'Pn of x' for any given 'n'. And we intuitively get the feeling that, as 'n' increases, 'Pn' hopefully will look more like 'f of x'. In other words, what we would like to do is to be able to compare, or better still, to study 'f of x' by comparing it with the limit of 'Pn of x' as 'n' goes to infinity.

In other words, what we hope will happen is that 'f of x' looks like ''P sub n' of x' for large values of 'n'. Now, we don't know where this is going to happen. In fact, the remainder of our course now revolves about three questions. The three questions are the following. First of all, does the limit, 'Pn of x' as 'n' approaches infinity, exist? In other words, does 'P of x' exist in the first place? Does this limit always exists for a given 'x'?

Secondly, suppose the limit does exist. How do we know it's going to equal the given function, 'f of x', which it's trying to approximate? In other words, how do we know that we don't get an approximation which is nice near the point of contact, but then, no matter how far out we go beyond the point of contact, the approximation doesn't become sufficiently good? See?

And the third question that comes up, assuming that the first have been answered, is suppose the limit does exist-- in other words, suppose this limit function, 'P of x', which is a limit of 'Pn of x' as 'n' approaches infinity, does exist-- the question is, does 'P' possess the polynomial properties that each piece of 'n' possesses? In other words, the question is, does the limit of a sequence of functions have the same property that each of the members of the sequence has?

For example, every polynomial is continuous. The question, for example, that might come up is, is the limit of a sequence of continuous functions also continuous? Now, at first glance, it might seem as if these are rather trivial to answer. Let me proceed next by showing that the answer can be no to each of these three questions, after which we'll then try to show when the answer will be yes.

The idea is this. Let me call these things counter-examples, even though I don't know what better word to use here. In other words, let me show you an example where the answer to each of questions one, two, and three happen to be false. For example, let 'f of x' be '1 over '1 minus x''. As a review in taking derivatives, 'f prime of x' would be '1 minus x' to the minus 2 power, ''f double prime' of x' would be 2 times '1 minus x' to the minus 3 power. See, in other words, you bring down the minus 2, but the derivative of what's inside with respect to 'x' is minus. So a minus times minus is plus.

Similarly, the third derivative of 'f of x' is 3 factorial, 2 times minus 3 times minus 1, '1 minus x' to the minus fourth. And in general, the n-th derivative of 'f', in this case, is 'n factorial' times '1 minus x' to the what? Minus 'n minus 1'. In other words, notice that the magnitude of the exponent is one greater than the number we're taking the factorial of. At any rate, if I now compute the n-th derivative of 'f' evaluated at 0, I just get 'n factorial'. And if I divide 'n factorial' by 'n factorial', I get 1. In other words, every one of my coefficients of the approximating sequence of polynomials is going to be 1.

In other words, the approximation for 'f of x', where 'f of x' is '1 over '1 minus x'', is simply what? 'Pn of x' is '1 plus x plus 'x squared'', et cetera, plus et cetera, 'x to the n-th' power. Now, here's the interesting point. This part is irrelevant. I just wanted to show you something over here. Notice that 'f of 2' is clearly minus 1. If I replace 'x' by 2 over here, 1 over 1 minus 2 is minus 1.

On the other hand, what I claim is that the limit of a sequence, ''P sub n' of x', as 'n' goes to infinity doesn't even exist when 'x' is 2, because look what happens over here. When 'x' is 2, this is what? 1 plus 2 plus 2 squared plus et cetera '2 to the n'. That's 1 plus 2 plus 4 plus 8 plus 16, et cetera. And as I let 'n' go to infinity, that sum increases without bound. In other words, here's an example where the given limit didn't have to exist. In other words, the sequence of polynomials does not converge at all at 'x' equals 2.

So much for showing that the answer to the first question can be no. As for the second question, let's define a function as follows, to show you why continuity is so important. Let 'f of x' be 'x squared' if 'x' is less than or equal to 2. And let it be 4 if 'x' is greater than 2. Now, if we compute the derivatives of 'f' evaluated at 0, we find that what? 'f of 0' is 0. See, 'x squared' at

0 is 0. '2x' at 0 is 0. The second derivative here is 2, so ''f double prime' of 0' is 2. And the third derivative and higher, since this is only 'x squared', are identically 0.

In other words, notice that the second derivative of 'f' evaluated at 0 is 2. All other derivatives are 0. In other words, notice that 'Pn of x' is equal to 'x squared' when 'n' is greater than 2. And that's a straightforward computation. I will drill you on the homework problems to show you how to get more facility. We're doing these things.

But here's the key point. Let's look to see what the limit of 'Pn of x' is as 'n' approaches infinity. Since 'Pn of x' is equal to 'x squared' for all 'n', in particular, then, since this doesn't depend on 'n', the limit off 'Pn of x' as 'n' goes to infinity is just 'x squared'. What this says is this. Our function, 'f of x', graphs like this. The sequence of polynomials-- which, by the way, converge to the limit function, 'x squared'-- look what they do. They're just 'y' equals 'x squared'.

In other words, somehow or other, when we got to this sharp corner-- in other words, notice that 'f of x' is not differentiable at this point-- our sequence of polynomials blissfully went right on their merry way smoothly while the function itself leveled off like this. In other words, the limit function always exist, but as soon as 'x' is greater than 2, in this example, the function, 'P of x', no longer is the same as 'f of x'. You see, 'P of x' is going up like this, 'f of x' has just leveled off this way.

There are even more glaring examples, but I will leave those for the supplementary notes to go into in more detail. But so far, we've seen what? That the answers to questions one and two can be no. Let's now show that the answer to question three can also be no.

Let's define a sequence of polynomials, ''P sub n' of x', to be 'x to the n', where the domain of 'P sub n' is the interval from 0 to 1. What that means is something like this. See, here's 0, 0. And I hope this reminds you of a homework problem that we had very early in the course, but that's irrelevant whether it does or not. 'P1 of x' would look like this. 'P2 of x' would look like this. 'P3 of x' would look like this.

In other words, all of these curves pass through 0, 0 and 1, 1. But as 'n' increases, the degree of contact here gets better. At any rate, let's take a look at the limit function. 'P of x' is the limit of ''P sub n' of x' as 'n' approaches infinity. 'x to the n', as 'n' approaches infinity, well, if 'x' is less than 1 in magnitude, 'x to the n-th' approach is 0. On the other hand, if 'x' is equal to 1, 1 to the n-th power is still 1 as 'n' approaches infinity. Therefore, 'P of x' would be what? It's 0 if 'x' is greater than or equal to 0 but less than 1.

And it's 1 if 'x' equals 1. Now, here's the key point. Observe that, since 'Pn of x' is a polynomial, it's, in particular, continuous. In other words, each 'P sub n' is a continuous function, unbroken. On the other hand, look at the limit function. It's 0 all the way up to 1. And that 1, the function jumps to 1.

In other words, graph-wise, the thing looks like this. This is a graph, 'y' equals 'P of x'. It's 0. All of a sudden, it jumps. Is the limit function, therefore, continuous with 'x' equals 1. And the answer is no. There's a jump discontinuity here.

In other words, observe that each 'P sub n' is continuous when 'x' equals 1. The limit function exists, but it isn't continuous when 'x' equals 1. In other words, the properties of a sequence of convergent functions do not have to be inherited by the limit function.

That shows, then, that the answer to our three questions can be no. What we would like to do is find out when and if the answer will be yes. What we shall do for the remainder of today's lesson is answer the first two questions. And this, by the way, is also done very nicely in the text. The third question, for some reason or other, is beyond the scope of the text. We will discuss this in our future lectures, plus the supplementary notes. In fact, that the answer to question number three will be the end of our course, when we finally answer that particular question.

But at any rate, what I'm saying now is let's spend the remainder of today's lesson in showing how to answer questions one and two in the affirmative, and we'll save question three for the remaining lectures. The first thing I'd like to point out is the role of the ratio test and absolute convergence. Namely, given the series summation 'a n 'x to the n''-- And by the way, if you haven't noticed this by now, it's rather conventional sometimes, to save time, not to put the subscripts on here. But if it bothers you, I mean, I could do things like this. It's just that to save time and space, I sometimes will not put the subscripts on here. But at any rate, let's just talk about this for a moment.

Notice that a series converges as soon as it converges absolutely. The point is that, for a positive series, I can use the ratio test or its equivalent. Namely, for example, what I can do is to test summation absolute value, 'a n 'x to the n'', for convergence by the ratio test, by the comparison m by the integral test, et cetera. And what I'm saying is, if that particular series converges, then the original series, namely, the one without the absolute value signs, converges absolutely for that same value of 'x'.

Well, again, instead of talking about that, let's look at a particular example. And it looks like I've scraped the blackboard here a little bit. Let me just-- The first example is this. Let's suppose you look at summation n goes from 0 to infinity, absolute value of 'x to the n-th' power. By the way, notice that this is a geometric series with a ratio absolute value of 'x'. Consequently, the series converges if and only if the absolute value of 'x' is less than 1.

By the way, let me make a little aside here. Many people would tackle this problem by the ratio test, as I hinted at over here. Namely, let the n-th term be the absolute value of 'x to the n-th' power. Then the 'n plus first' term would be the absolute value of 'x to the 'n plus first'' power. Then the ratio between these two would be the absolute value of 'x'. Therefore, row, which is this limit, would be the limit of the absolute value of 'x' as 'n' goes to infinity.

That's just equal to the absolute value of 'x' itself. And for convergence, notice that row must be less than 1, and that says what? The absolute value of 'x' must be less than 1, just as we did over here. I simply would like to make a little aside over here, and that is, if you use this particular approach, that comes under the heading of circular reasoning, because you may recall when we proved the ratio test, we did it by comparing the series with the geometric series, which meant that we already had to know that the geometric series converged. And this is a geometric series.

But that's an aside which I just wanted to mention to you in forms of circular reasoning. The important point to observe is what? That this series converges if the absolute value of 'x' is less than 1. Consequently, that means that the series without the absolute value sign converges, in fact, absolutely, if the absolute value of 'x' is less than 1. In other words, for this particular series, '1 plus x plus 'x squared' plus 'x cubed'', et cetera, that will converge to a finite limit function if the absolute value of 'x' is less than 1.

Let's look at another example. Suppose we look at the absolute value of 'x to the n-th' power over 'n factorial'. If we look at this, the nth term is the absolute value of 'x to the n-th' over 'n factorial'. The 'n plus first' term, therefore, is the absolute value of 'x to the 'n plus 1'', over ''n plus 1' factorial'. Therefore, the ratio between the 'n plus first' term and the n-th term is the absolute value of 'x' over 'n plus 1'. Rho is this particular limit.

Well, the limit of the absolute value of 'x' over 'n plus 1', as 'n' approaches infinity-- and here's the key point. Notice that 'x' is a fixed but finite number once we get started here. In other words, we pick an 'x' and fixed it. That means that that stays constant. But as 'n' goes to infinity, the denominator increases without bound. That makes the limit equal to 0. And 0 is obviously less than 1, regardless of what the value of 'x' happens to be here.

In other words, in this particular case, this particular series converges for all real values of 'x'. And we sometimes abbreviate that by saying the absolute value of 'x' is less than infinity. And therefore, the original series, meaning the one without the absolute value sign, would, in particular, converge also for all real 'x'. The important point is that, in general, given any series-- in other words, limit of a sequence of polynomials of this particular type-- there always exists a number 'M'.

Now, 'M' may be as small as 0. It may be as large as infinity. But there always exists an 'M', such that the particular series will converge absolutely if the magnitude of 'x' is less than 'M', and diverge if the magnitude of 'x' is greater than 'M'.

In other words, this particular 'M', which is called the radius of convergence, governs the answer to the first question. In other words, it's this 'M', which we often find by the ratio test, that determines where the sequence of functions, ''P sub n' of x', for what values of 'x', that will converge to 'P of x'. And again, I think this is kind of difficult for you to see abstractly. The proof is done in the text. What I'd like to do is show you pictorially what's happening over here.

See, let's suppose we have our series, summation 'a n 'x to the n''. And we pick a particular value of 'x', say 'x sub 1'. Now, the idea is this. If this particular series converges-- and the proof is given in the text, and I won't repeat it here because I want you to see the overview here-- if this particular series converges-- in other words, if the series converges when 'x' is equal to 'x sub 1'-- then it can be shown that it converges absolutely for any 'x' which is smaller in magnitude then 'x sub 1'.

In other words, the gist is this. If I know that the series converges over here, I can then draw this interval surrounding 0, and conclude that the series converges in this entire interval. Correspondingly, if this particular series diverges-- and by the way, the whole proof of this thing hinges on the comparison test, essentially. See, if this thing diverges, then certainly, if 'x' is larger than 'x1' in magnitude, this will also diverge. In other words, once I know that my series diverges at this particular value of 'x1'.. of 'x', I know it diverges for all values of 'x' which are greater in magnitude then 'x1', which means, in terms of a picture, for everything further away from 0, then 'x1' and minus 'x1'.

In other words, if this converges, this converges. In here, if this diverges, we have divergence out here. Now, what does that mean? Let me show you again in terms of a extended diagram. See, the idea goes something like this. Given the series summation 'a n 'x to the n'', pick a number 'x' equal to 'x1'. Suppose the series converges at 'x1'. Then we know that it converges every place in here, so there's no need to check this any further.

What we do next is we pick a number, 'x2', outside of this interval. Say 'x2' is over here. Suppose, for the sake of argument-- see, what we're saying is 'a n 'x 1 to the n'' converges, so you just suppose this. That gives us this picture. Suppose 'a n 'x2 to the n'' diverges. Then we know that the series diverges for all larger values of 'x'.

Now we know how the series behaves exactly, except in the interval between 'x1' and 'x2'. So what we do next is what? We pick a number which we'll call 'x3'. We then see whether this converges or diverges. If it converges at 'x3', we know that the radius of convergence extends all the way between minus 'x3' and plus 'x3'. If it diverges, we know that, from 'x3' out and minus 'x3' out, the series diverges.

In any event, each time we perform this operation, we cut the remaining space down. And we can keep on going this way. And you see what you're doing then, is you're zeroing in on this number that we called capital 'M' before. You see, it's just a series of refinements. And by the way, the formal proof of this looks very messy. And yet, when it's stripped of all embellishment, that's exactly what the formal proof says.

Well, at any rate, I will reinforce the rest of this between the text, the notes, and the learning exercises. All I want to do now is talk about the second question. In other words, we've now talked about something called the radius of convergence, that tells us when the series of functions converges to the limit function. Now the question is, how can we measure whether that limit function converges to the original function that we're trying to approximate?

In other words, how do we know whether 'P of x' actually equals 'f of x'? And again, this is done extremely well in the text. It's called Taylor's theorem with remainder. The proof is given very, very nicely. I think the hardest part is to show what the thing means.

You see, using integration by parts repeatedly-- which is done in the text-- it follows that, if 'f' and its first 'n plus 1' derivatives exist at 'x' equals 0, it turns out-- and this is why this is called Taylor's theorem with remainder-- it turns out that 'f of x' can be written as 'Pn of x' plus a remainder term-- ''r sub n' of x'-- a remainder term. And then you see, computationally, the text shows what that remainder term looks like. It happens to be the integral from 0 to 'x', ''x minus t' to the n-th' over 'n factorial', 'n plus first' derivative of 'f of t', 'dt', from 't' equals 0 to 't' equals 'x'.

This looks messy. It is messy. There's no need to hammer this home in the lecture part of our course. What I prefer to do is to give several learning exercises that will give you drill in using this. But the point that I want you to understand is that, since this is all done well in the text, all I want you to do is to see where this thing fits in. In other words, what the text is showing is, look it, the difference between the function which we're approximating at 0 comma 'f of 0' by this sequence of polynomials, the difference between that function and the nth member of that sequence of polynomials is simply something called ''r sub n' of x', where numerically ''r sub n' of x' looks like this.

And the point is that, if you now take the limit of this expression as 'n' goes to infinity, remember, 'P of x' is this limit. Therefore, the only way that 'P of x' can equal 'f of x' is if, for a given value of 'x', this remainder goes to 0 as 'n' goes to infinity. In other words, leaving the computational details to the text, because it does it very nicely, what we're saying is that the significance of Taylor's remainder theorem is that, to find out where 'f of x' and 'P of x' are identical, we simply-- and I say simply meaning conceptually simply-- computationally, it might be difficult.

What we do computationally is simply find all those values of 'x', for which ''r sub n' of x' goes to 0 in the limit as 'n' goes to infinity. You see, what we've done is we've now done two things. We've first shown when the sequence of polynomials has a limit. And secondly, we've shown when it does have a limit, when will it equal the given function that it's trying to represent. The question that still remains is, how do we know that the limit function has the same properties as each member of the sequence?

In fact, we've just given an example where the limit function had different properties. And the question is, under what conditions can we be sure that certain nice results-- that we'd like to be true, because they're true about each polynomial in the sequence-- actually are true about the limit function. In fact, that shall be our topic for the next two lessons. And these next two lectures should complete the course. At any rate, until next time, goodbye.

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