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PROFESSOR:

Hi. Our lecture for today probably should be entitled it should've been functions, but it's analytic geometry instead, or a picture is worth a thousand words. What we hope to do today is to establish the fact that whereas in the study of calculus when we deal with rate of change we are interested in analytical terms, that more often than not, we prefer to visualize things more intuitively in terms of a graph or other suitable visual aid, and that actually, this is not quite as alien or as profound as it may at first glance seem.

Consider, for example, the businessman who says profits rose. Profits rose. Now, you know, profits don't rise unless the safe blows up or something like this. What profits do is they increase or they decrease.

The reason that we say profits rise is that when the profits are increasing, if we are plotting profit in terms of time, the resulting graph shows a rising tendency. As the profit increases, the curve rises. And in other words then, we begin to establish the feeling that we can identify the analytic term increasing with the geometric term rising. And this identification, whereby difficult arithmetic concepts are visualized pictorially, is something that begins not only very early in the history of man, but very early in the development of the mathematics curriculum.

Oh, as a case in point, consider the problem of 5 divided by 3 versus 6 divided by 3. I remember when I was in grade school that this particular problem always seemed more appealing to me than this problem, that 6 divided by 3 seemed natural, but 5 divided by 3 didn't. And the reason was is that in terms of visualizing tally marks, it was much easier to see how you divide six tallies into three groups than five tallies into three groups.

However, as soon as we pick a length as our model, the idea is this. Either one can divide a line into three parts of equal length or one can't divide the line. Now, if I can geometrically divide this line into three equal parts, and in plain geometry we learn to do this, then the fact is that I can divide this line segment into three equal parts regardless of how long this line happens to be.

Oh, to be sure, if this line happens to be 6 units long, this point is named 2. And if, on the other hand, the line happens to be only 5 inches long, the resulting point is named $5/3$. But notice that in either case, I can in a very natural way define or identify either ratio as a point on the line. And this idea of identifying numerical concepts called numbers with geometric concepts called points is a very old device and a device that was used and still is used in the curriculum today under the name of the number line, under the name of graphs, and what we will use as a fundamental building block as our course develops.

Now, you know, in the same way that we can think of a single number as being a point on the line, we can think of a pair of numbers, an ordered pair of numbers, as being a point in the plane. This is Descartes geometry, which we can call coordinate geometry, the idea being that in the same way as we can locate a number of along the x-axis, shall we say, we could have located a numbered pair as a point in the plane. Namely, 2 comma 3 would mean the point whose x-coordinate was 2 and whose y-coordinate was 3.

By the way, the reason that we say ordered pairs is, if you observe, 2 comma 3 and 3 comma 2 happen to be different pairs. And notice again how vividly the geometric interpretation of this is. Namely, in terms of locating a point in space, it's obvious that the point named 2 comma 3 is not the same as the point named 3 comma 2. Again, the important thing is this. When I think of 5 divided by 3, when I think of the ordered pair 2 comma 3, I do not have to think of a picture. I can think of these things analytically. But the picture gives me certain insights that will help me with my intuition, an aid that I don't want to relinquish.

For example, going back to the graph again. Thinking of the analytic term greater than, notice how much easier it is to think of, for example, higher than, see, one point being higher than another or to the right of. You see, geometric concepts to name analytic statements, or instead of increasing, as we mentioned before, to say rising. And there will be many, many more such identifications as we go along with our course.

At any rate, let's continue to see then what is the relationship then between functions that we talked about and graphs? The idea is something like this. Let's return to our friend of the first lecture: s equals $16t$ squared.

We can think of a distance machine being the function where the input will be time and the output will be what? The square of the input multiplied by 16. Observe that from this, I do not have to have any picture to understand what's happening here. Namely, I can measure an

input, measure an output, and observe analytically what is happening. However, as we saw last time, our graph sort of shows us at a glance what seems to be happening, that we can identify rising and falling with increasing and decreasing and things of this type. We will explore this, of course, in much more detail as we continue in our course.

By the way, there's no reason why the input has to be a single number. For example, why couldn't the input be an ordered pair of numbers? Among other things, let's take a simple geometric example. Consider, for example, finding the volume of a cylinder in terms of the radius of its base and the height. We know from solid geometry that the volume is $\pi r^2 h$. We could therefore think of a volume machine where the input is the ordered pair r, h , and the output is the single number $\pi r^2 h$.

By the way, notice here the meaning of ordered pair. You see, if the pair $2, 3$ goes into the machine, notice that the recipe here says what? You square the first member of the pair. In other words, if $2, 3$ is the input, we square 2, which is 4, multiplied by 3, which is 12, and 12 times π , of course, is 12π . On the other hand, if the input is $3, 2$, the first number is 3. Our recipe squares the first number. That would be 9, times 2 is 18, times π is 18π .

But again, observe that I at no time needed a picture to visualize what was happening here. Of course, if I wanted a picture, I could try to plot this also, but notice now that my graph would probably need three dimensions to draw. And why would it need three dimensions? Well, notice that my input has two independent measurements r and h , and therefore, I would need two dimensions just to take care of r and h . Then I would need a third dimension to plot v .

And by the way, notice the next stage. If I had an input that consisted of three independent measurements, this would still make sense, but now I would be at a loss for the picture. In other words, what I'm trying to bring out next is the fact that whereas pictures are a tremendous help, maybe a second subtitle to our lecture should've been a picture is worth a thousand words provided you can draw it. Because, you see, if we needed three independent dimensions to locate the input and then a fourth one to locate the output, how would we draw the picture?

By the way, this happens in high school algebra again, if you want to see the analogy. Look at, for example, the algebraic equation $a + b^2$. We learned in algebra that this is $a^2 + 2ab + b^2$. Observe that we do not need to have a picture to

understand how this works. Oh, to be sure, if we had a picture, we get a tremendous amount of insight as to what's happening here. Namely, let's visualize a square whose side is $a + b$.

On the one hand, you see, the area of the square would be $(a + b)^2$, you see, the side squared. On the other hand, if we now subdivide this figure this way, we see that that same square is made up of four pieces having one piece of area a^2 , two pieces of area ab , and one piece of area b^2 . And so we see, on the other hand, that the area of the square is $a^2 + 2ab + b^2$. And so again, observe that whereas this stands on its own two legs, the picture helps us quite a bit.

By the way, we could continue this with a cube over here. Namely, it turns out that $(a + b)^3$ is $a^3 + 3a^2b + 3ab^2 + b^3$. And again, if we wanted to, and we won't take the time here, but if we wanted to, we could now draw a cube whose side is $a + b$. Namely, we could take the same diagram that we had before and now make a third dimension to it. And if you did that, you would see what? That the cube whose volume is $(a + b)^3$ is divided into eight pieces, one of size a by a by a , three of size a by a by b , three of size a by b by b , and one of size b by b by b . Again, the picture is a tremendous visual aid.

Now, the key step is this. If we were to now give in to our geometric intuition and say, lookit, why worry about algebra when it's so much easier to do this thing by geometry, the counterexample is consider $(a + b)^4$. Now, by the binomial theorem, and I might add, the same binomial theorem that allowed us to get these results, we can also write that this is what? It's $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$.

Now, it's not important how we get this result. The important point is that analytically, we can raise a number to the fourth power just as easily as we can to the third power or the second power. The only difference is that in the case of the third power, we had a picture that we could use. In the fourth power case, we didn't have a picture. And what a tragedy it would have been to say, hey, we can't solve this problem because we can't draw the picture.

And by the way, as a rather interesting aside, notice the geometric influence on how we read this. This is called $(a + b)^4$. But somehow or other, we don't call this one $(a + b)^3$. We call it $(a + b)^3$, suggesting the geometric configuration of the cube. And here we don't usually say $(a + b)^2$. We say $(a + b)^2$.

squared.

You see, the idea is that when the picture is available, it gives us a tremendous insight as to what can be done, and it helps us learn to visualize what's happening analytically. In fact, what usually happens is we use the picture to justify what's happening analytically when we can see the picture and then just carry the analytic part through unimpeded in the case where we can't draw the picture.

Let me give you another example of this. Let's look at the following set. That also should review our language of sets for us. Let S be the set of all ordered pairs x comma y such that x squared plus y squared equals 25. Question: Does the ordered pair 3 comma 4 belong to S ? Answer: Yes. How do we know?

Well, we have a test for membership. We're supposed to do what? Square each of the entries, each of the numbers, add them, and if the answer is 25, then that ordered pair belongs to S . 3 squared plus 4 squared is 25, so this pair belongs to S . How about 1 comma 2? Well, 1 squared plus 2 squared is 1 plus 4, which is 5. 5 is not equal to 25, so 1 comma 2 does not belong to S .

Well, did we need any geometry to be able to visualize this result? Hopefully, one did not need any geometry to visualize this result. On the other hand then, what does it mean in analytic geometry when we say that x squared plus y squared equals 25 is a circle? As badly as I draw, x squared plus y squared equals 25 looks less like a circle than the circle I drew over here.

You see, what we really mean is this. Consider all the points in the plane x comma y for which x squared plus y squared equals 25. These are precisely the points on this particular circle. And the easiest way to see that, of course, is that since the radius of the circle is 5 and the point x comma y means that this length is x and this length is y , notice that from the Pythagorean theorem, we see it once, that x squared plus y squared equals 25.

Now again, the solution set to this equation is our set S whether we're thinking of this thing algebraically or geometrically. On the other hand, watch what our picture seems to give us that we didn't have before. Let's return to our point 1 comma 2, which we saw didn't belong to S . Well, look at this. Where would 1 comma 2 be? 1 comma 2 would be inside the circle.

Why is that? Because, you see, if we take the point 1 comma 2, if we take, say, for example,

the point 1 comma 2, notice that the distance from the origin to the point 1 comma 2 is less than 5. In other words, the distance is less than 5 so the square of the distance is less than 25. In other words, not only can we say that 1 comma 2 does not belong to S, which we could have said without the picture, we can now say what? 1 comma 2 is-- and notice the geometric language here-- is inside the circle.

In other words, the study of inequalities can now be reduced. Instead of talking about less than and greater than, we can now talk about such things as inside and outside. You see, inside the circle, which is a simple geometric concept, just means what? A set of all points for which $x^2 + y^2$ is less than 25. Outside that circle, $x^2 + y^2$ is greater than 25. On the circle, $x^2 + y^2$ equals 25. Again, a nice identification between numbers and pictures, analysis and geometry.

Well, this then shows us why we want to study pictures rather than functions. Now, if we look at any textbook in which we deal with graphs, it always seems that we start with graphs of straight lines. And the question is what is so great about a straight line? After all, pictures in general are going to be much more complicated than that. What advantage is there in starting with straight lines? And again, we begin to realize how straight lines are the backbone of all types of analytical procedures and all types of curve plotting.

For example, let's suppose we were studying this particular curve, and we wanted to know what was happening to that curve in the neighborhood around the point p. Let's draw in the tangent line to the curve at the point p. Notice that this line that we've drawn serves as a wonderful approximation curve itself if we stay close enough to the point of tangency. In other words, notice how much we can deduce about this curve if we study only the straight line segment at the point of tangency. Of course, the approximation gets worse and worse as we move further and further out. But in the neighborhood of the point of what's going on, notice again then that the straight line is an important building block.

By the way, again we use straight lines in a rather subtle way in something called interpolation. For example, let's suppose I go to a log table and I look up the log of 2. I find that the log of 2 is 0.301. I look up the log of 4. I find that's 0.602. Now I look for the log of 3, and I see that it's been obliterated. I don't know what it is. So I say, well, let me guess. 3 is halfway between 2 and 4. Therefore, I would suspect that the log of 3 is halfway between the log of 2 and the log of 4. And so, halfway between here would be about what? 0.452, roughly speaking. All of a sudden, the obliteration on my book clears up. And I look, and I don't find 0.452. Instead I find-

- well, let's write it over here. What I find is 0.477.

Now, you know, this is a pretty big error to attribute to slide rule inaccuracy or trouble in rounding off the tables. What really went wrong over here? And the answer comes up again that unless otherwise specified, the process known as interpolation hinges on replacing a curve by a straight line approximation. In fact, you see, if we were to draw the curve of the logarithm function, we would find that the picture is something like this. And when we looked up the log of 2, this height is what we found in the table. When we looked up the log of 4, this height is what we would have found in the table. If we had looked up in the table the log of 3, this is the height that we would have found. This is the height that's 0.477.

Notice that in general, if we go halfway from here to here, we do not go halfway from here to here. It depends on the shape of the curve. The only time you can be sure that you have proportional parts is if the curve that joined these two points was a straight line. And notice, by the way, that by the shape of this curve, the straight line falls below the curve, and therefore, the height that we found was to the straight line, not to the curve. That was the point 0.452.

So, in other words, notice that we got smaller than the right answer because we approximated as if it was a straight line that was joining the curve. You see, what interpolation hinges on is that the size of the interval is very small and that you can assume that for the accuracy that you're interested in that the straight line approximation to the curve is sufficiently accurate to represent the curve itself.

Well, enough about that. Once we've talked about why straight lines are important, the next thing is how do we measure straight lines? See, another interesting point to something like this. Many times we know what something means subjectively, but we don't know what it means objectively. For example, one way of finding a line is to know two points on the line.

Another way is to know one point and the slant of the line. The question that comes up is how do you measure the slant of a line? In other words, shall you say that the line is very slanty? And if the answer to that is yes, how do you distinguish between slanty and very slanty, steep and very steep, very steep and very, very steep? We need something more objective. And the way we get around this is as follows.

Given a line, we define the slope as follows. We pick any two points on the line. And from those two points, we can measure what we call the run of the line, in other words, how far you've gone this way, and the rise of the line, how much it's risen this way. And what we do is

we define the slope to be the rise divided by the run, or without the delta notation in here, y_2 minus y_1 over x_2 minus x_1 .

By the way, there are little problems that come up. After all, our answer should not depend on the picture. It should be sort of self-contained analytically. The question comes up is what if I had labeled this point x_2, y_2 and this point x_1, y_1 ? What would have happened then? And observe that as long as we keep the pairs straight, it makes no difference whether you write this or whether you write this. Because, you see, in each case, all you've done is change the sign, and negative over negative is positive. So certainly our answer to what a slope is is objective enough, so it does not depend on how the points are labeled.

A second objection that most people have is they say something like, well, who are you to say that we pick these two points? What if I came along and picked these two points and I now computed the slope by taking this as my delta y and this as my delta x ? Obviously, it would be a tragedy if the answer to the problem depended on which pair of points you picked since a line should have but one slope.

Again, notice that our high school training in geometry, similar triangles, motivates why we pick ratios. Namely, while this delta y and this delta y may be different and this delta x and this delta x may be different, what is true is that the ratio of this delta y to this delta x is the same as the ratio of this delta y to this delta x . And that's why we pick the ratio.

By the way, another way of talking about ratio is if you look at delta y divided by delta x and you've had some trigonometry, it reminds you of a trigonometric relationship. Namely, you look at delta y , you look at delta x , and you say, my, isn't that just the tangent of this particular angle? Couldn't I define the slope? And by the way, the general symbol for slope, for better or for worse, just happens to be letter m . Why couldn't I define m to be the tangent of ϕ , where ϕ is the angle that the straight line makes with the positive x -axis?

And, of course, there is a little subtlety here that we should pay attention to. This would be an ambiguous definition if the scale on the x - and the y -axis were not the same. In other words, notice that by changing the scale here, I can distort the same analytic information. So if I agree, however, that the unit on the x -axis is the same as the unit on the y -axis, then I can say, OK, the slope is also tangent of the angle ϕ . I much prefer to say it's delta y divided by delta x , because then if I forget the scale, I'm still in no trouble.

On the other hand, if we use the tangent definition, we can utilize all we know about trigonometry to get some other interesting results. Namely, the question that might come up is can we study the slopes of two different lines very conveniently in terms of our definition of slope? And the answer is this. If we imagine now that our lines are drawn to scale here, and here are two different lines, which I'll call l_1 and l_2 , and we'll call the angle that l_1 makes with the positive x-axis ϕ_1 , the angle that l_2 makes with the positive x-axis ϕ_2 . Therefore, what? m_1 is $\tan \phi_1$. m_2 is $\tan \phi_2$.

Notice that our formula for the tangent of the difference of two angles-- you see, notice that this angle here is what? Since this angle is the sum of these two, this angle here is ϕ_2 minus ϕ_1 or the negative of ϕ_1 minus ϕ_2 . I should have had this ϕ_2 minus ϕ_1 , but since that just changes the sign, that will not have any bearing on the point I want to make.

Let's continue this way. Tangent of ϕ_1 minus ϕ_2 is $\tan \phi_1$ minus $\tan \phi_2$ over 1 plus $\tan \phi_1 \tan \phi_2$. On the other hand, by our definitions of m_1 and m_2 , this is m_1 minus m_2 over 1 plus $m_1 m_2$.

Now, this tells me how to find the angle between two lines just in terms of knowing the slope. Two very special interesting cases as extremes suggest themselves right away. One case is what happens if the lines are parallel? If the lines are parallel, you see, ϕ_1 equals ϕ_2 , in which case ϕ_1 minus ϕ_2 is 0 , in which case the tangent of ϕ_1 minus ϕ_2 had better be 0 . But the only way a fraction can be 0 is for the numerator to be 0 , and that means that m_1 must equal m_2 . In other words, in terms of slopes, we can study parallel lines just by equating their slopes.

A less obvious relationship that's equally important is how do you measure whether two lines are perpendicular? The answer is if they're perpendicular, the angle between them is 90 degrees. The tangent of 90 degrees is infinity, as we learned. That's equivalent to saying what? That the denominator is 0 . See, the only way a fraction blows up is for the denominator to be 0 . But the only way that 1 plus $m_1 m_2$ can be 0 is for what? $m_1 m_2$ to be equal to minus 1 . And this gives us the other very well-known result that in terms of slopes, to study whether two lines are perpendicular, all we need to investigate is whether one slope is the negative reciprocal of the other.

Well, again, the textbook will bring out slopes in more detail. The next question that we'd like to bring up in terms of a picture is worth a thousand words is how do you identify the straight line

with an algebraic equation? What do we mean by the equation of a straight line? Well, again, there are two possibilities.

The first possibility is that the line is parallel to the y -axis. If the line is parallel to the y -axis, if the line goes through the point a comma 0 , notice that the only criteria for the point to be on that line is that its x -coordinate equal a . By the way, this is often abbreviated in the textbook as the line x equals a . Many a student says how do you know this is a line? Why isn't this the point x equals a ?

And here again is a good review of why we stress the language of sets. Here again is a good reason why we express the language of sets so strongly. Namely, go back to the universe of discourse here. When you see the set of all ordered pairs x comma y for which x equals a , this gives you the hint that you're talking about pairs of points, and that tells you that you have numbers in the plane, not on the line, not on the x -axis, a two-dimensional interpretation over here.

You see, if this said the set of all x such that x equals a , it would just be a point. But notice the hint over here. At any rate, this then becomes the equation of a straight line if the straight line is parallel to the y -axis. Of course, the other possibility is what if the line isn't parallel to the x -axis?

And here, too, we say OK, suppose we know a point on the line and suppose we know the slope of the line. What we will do is pick any other point on the plane, which we will label arbitrarily x comma y , and see what the equation x comma y has to satisfy. How do the coordinates have to be related to be on this line? Well, we already know that slope does not depend on which two points you pick. Consequently, since the slope of this line is m , the slope must also be what? y minus y_1 over x minus x_1 . And this becomes the fundamental definition for the equation of a line which is not parallel to the y -axis.

And by the way, again, I think m 's and x_1 's and y_1 's tend to give you a bit of hardship at first until you get used to them. Let's illustrate this thing with a specific example. Suppose I say to you I am thinking of the line whose slope is 3 and which passes through the point 2 comma 5 . And notice the language of sets here. To say that 2 comma 5 is on the line is the same as saying that 2 comma 5 belongs to the set of points determined by the line. Drawing a rough sketch over here-- and by the way, notice something very important here. I never have to draw to scale. Because, you see, all I'm going to use is the analytic terms, and 2 comma 5 is still 2

and 5, no matter how I draw the picture.

So, for example, if I say OK, let's see what it means for the point x comma y to belong here, I say, well, what does that mean? My slope is going to have to be what? y minus 5. That's my rise. My run is x minus 2, and that must equal 3. And if I clear this of fractions, I get what? y is equal to $3x$ minus 1. By the way, does this check out? If x is 2, 2 times 3 is 6, minus 1 is 5. 2 comma 5 is on the line.

You see, here's the thing. We talked about the line geometrically. Now I have an algebraic equation. I no longer have to refer to the picture. I have something analytic now. For example, suppose a person says to me I wonder if the point 8 comma 23 is on this line? I don't have to draw a picture to scale. I don't have to waste any time. I know that the equation of my line is y equals $3x$ minus 1.

By the way, if y equals $3x$ minus 1, as soon as x is 8, what must y equal? y must equal what? 23? Is that right? And so is the point 8 comma 23 on the line? Yes. How about 8 comma 12? 8 comma 12 isn't on the line because 3 times 8 minus 1 is not 12. But notice that we can even see algebraically that 8 comma 12 must be below the line.

In other words, our study of equations allows us not only to visualize lines as equations, but we can also visualize inequalities as pictures. In other words, if we have the equation of a line, if this is the line y equals, say, $3x$ plus 1 or something like this, then what is this region here? These are all those values which lie-- whose heights lie below the height to be on the curve. Again, not a very clear example in the sense of drawing the picture neatly for you, but our main aim is not to draw neat pictures here. Our main aim is to show how analytical terms can be studied very conveniently in terms of pictures.

In fact, perhaps to conclude today's lesson, what we should talk about is an old algebraic concept called simultaneous equations. Suppose you're asked to solve this pair of equations. You say, well, let's see, if y equals $3x$ minus 1 and it's also equal to x plus 1, that says that x plus 1 equals $3x$ minus 1. I now solve this thing algebraically. I get $2x$ equals 2, so x equals 1. Knowing that x equals 1, I can see that y equals 2, and I see that 1 comma 2 is my solution. In other words, if I wound up with this thing algebraically, 1 comma 2 is the only member that belongs to both of these two solution sets.

Now, again, notice how I can solve this purely analytic problem without recourse to a picture. On the other hand, if I want to think of this thing pictorially, notice that y equals $3x$ minus 1 is

the equation of a particular straight line, and y equals x plus 1 is also the equation of a line. Notice that since these two lines are not parallel, they intersect at one particular point.

And the geometric problem that I solved on the previous board turns out to be that the point $(1, 2)$ is the point that both of these lines have in common. In fact, if we call this line as before L_1 and if we name this L_2 , in the language of sets, $(1, 2)$ is what? The point which is the intersection of the two lines L_1 and L_2 , again, a geometric interpretation for an analytic problem.

In fact, notice also how much mileage I can get out of the geometric picture. For example, notice that this region here has a very nice geometric interpretation. It's the set of what? All points which are below this line and above this line. In other words, what? To be below this line, y must be less than x plus 1, and to be above this line, y must be greater than $3x$ minus 1.

Notice then that a pair of simultaneous inequalities, which may not be that easy to handle, are very easy to handle in terms of regions in the plane. Notice also that since two lines can either be parallel or not parallel, we also get a nice geometric interpretation as to why simultaneous equations may have one solution. Namely, the lines are not parallel, and hence, they intersect. Or no solutions, the lines could've been parallel without intersecting. Or infinitely many solutions, the two lines could have been different equations. In effect, I should say what? The two equations could've been different equations for the same line.

Again, this may seem a little bit sketchy and rapid, but all we want to do is give the overview. The reading assignment in the text goes into great detail on the points that we've mentioned so far. But again, in summary, what our lesson was supposed to be today was to indicate the importance of being able to visualize and to identify analytic results with geometric pictures. And so, until next time, goodbye.

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