**61** For any power $n$, Problem 6.2.59 proved $e^x > x^n$ for large $x$. Then by logarithms, $x > n \ln x$. Since $(\ln x)/x$ goes below $1/n$ and stays below, it converges to ________.

**62** Prove that $y \ln y$ approaches zero as $y \to 0$, by changing $y$ to $1/x$. Find the limit of $y^y$ (take its logarithm as $y \to 0$). What is $.1^{.1}$ on your calculator?

**63** Find the limit of $\ln x/\log_{10}x$ as $x \to \infty$.

**64** We know the integral $\int_1^x t^{h-1}\,dt = [t^h/h]_1^x = (x^h - 1)/h$. Its limit as $h \to 0$ is ________.

**65** Find linear approximations near $x = 0$ for $e^{-x}$ and $2^x$.

**66** The $x^3$ correction to $\ln(1+x)$ yields $x - \frac{1}{2}x^2 + \frac{1}{3}x^3$. Check that $\ln 1.01 \approx .0099503$ and find $\ln 1.02$.

**67** An ant crawls at 1 foot/second along a rubber band whose original length is 2 feet. The band is being stretched at 1 foot/second by pulling the other end. At what time $T$, *if ever*, does the ant reach the other end?

One approach: The band's length at time $t$ is $t + 2$. Let $y(t)$ be the *fraction* of that length which the ant has covered, and explain

(a) $y' = 1/(t+2)$ (b) $y = \ln(t+2) - \ln 2$ (c) $T = 2e - 2$.

**68** If the rubber band is stretched at 8 feet/second, when if ever does the same ant reach the other end?

**69** A weaker ant slows down to $2/(t+2)$ feet/second, so $y' = 2/(t+2)^2$. Show that the other end is never reached.

**70** The slope of $p = x^x$ comes two ways from $\ln p = x \ln x$:

**1** Logarithmic differentiation (LD): Compute $(\ln p)'$ and multiply by $p$.

**2** Exponential differentiation (ED): Write $x^x$ as $e^{x \ln x}$, take its derivative, and put back $x^x$.

**71** If $p = 2^x$ then $\ln p =$ ________. LD gives $p' = (p)(\ln p)' =$ ________. ED gives $p = e^{\_\_\_\_}$ and then $p' =$ ________.

**72** Compute $\ln 2$ by the trapezoidal rule and/or Simpson's rule, to get five correct decimals.

**73** Compute $\ln 10$ by either rule with $\Delta x = 1$, and compare with the value on your calculator.

**74** Estimate $1/\ln 90{,}000$, the fraction of numbers near 90,000 that are prime. (879 of the next 10,000 numbers are actually prime.)

**75** Find a pair of positive integers for which $x^y = y^x$. Show how to change this equation to $(\ln x)/x = (\ln y)/y$. So look for two points at the same height in Figure 6.13. Prove that you have discovered all the integer solutions.

***76** Show that $(\ln x)/x = (\ln y)/y$ is satisfied by

$$x = \left(\frac{t+1}{t}\right)^t \quad \text{and} \quad y = \left(\frac{t+1}{t}\right)^{t+1}$$

with $t \neq 0$. Graph those points to show the curve $x^y = y^x$. It crosses the line $y = x$ at $x =$ ________, where $t \to \infty$.

## 6.5 Separable Equations Including the Logistic Equation

This section begins with the integrals that solve two basic differential equations:

$$\frac{dy}{dt} = cy \qquad \text{and} \qquad \frac{dy}{dt} = cy + s. \tag{1}$$

*We already know the solutions.* What we don't know is how to discover those solutions, when a suggestion "*try* $e^{ct}$" has not been made. Many important equations, including these, separate into a $y$-integral and a $t$-integral. The answer comes directly from the two separate integrations. When a differential equation is reduced that far—to integrals that we know or can look up—it is solved.

One particular equation will be emphasized. The ***logistic equation*** describes the speedup and slowdown of growth. Its solution is an **S-*curve***, which starts slowly, rises quickly, and levels off. (The 1990's are near the middle of the S, if the prediction is correct for the world population.) S-curves are solutions to ***nonlinear*** equations, and we will be solving our first nonlinear model. It is highly important in biology and all life sciences.

## SEPARABLE EQUATIONS

The equations $dy/dt = cy$ and $dy/dt = cy + s$ (with constant source $s$) can be solved by a direct method. ***The idea is to separate y from t***:

$$\frac{dy}{y} = c\,dt \quad \text{and} \quad \frac{dy}{y + (s/c)} = c\,dt. \tag{2}$$

All $y$'s are on the left side. All $t$'s are on the right side (and $c$ can be on either side). This separation would not be possible for $dy/dt = y + t$.

Equation (2) contains differentials. They suggest integrals. The $t$-integrals give $ct$ and the $y$-integrals give logarithms:

$$\ln y = ct + \text{constant} \quad \text{and} \quad \ln\left(y + \frac{s}{c}\right) = ct + \text{constant}. \tag{3}$$

***The constant is determined by the initial condition.*** At $t = 0$ we require $y = y_0$, and the right constant will make that happen:

$$\ln y = ct + \ln y_0 \quad \text{and} \quad \ln\left(y + \frac{s}{c}\right) = ct + \ln\left(y_0 + \frac{s}{c}\right). \tag{4}$$

Then the final step isolates $y$. The goal is a formula for $y$ itself, not its logarithm, so take the exponential of both sides ($e^{\ln y}$ is $y$):

$$y = y_0 e^{ct} \quad \text{and} \quad y + \frac{s}{c} = \left(y_0 + \frac{s}{c}\right)e^{ct}. \tag{5}$$

It is wise to substitute $y$ back into the differential equation, as a check.

This is our fourth method for $y' = cy + s$. Method 1 assumed from the start that $y = Ae^{ct} + B$. Method 2 multiplied all inputs by their growth factors $e^{c(t-T)}$ and added up outputs. Method 3 solved for $y - y_\infty$. Method 4 is ***separation of variables*** (and all methods give the same answer). This separation method is so useful that we repeat its main idea, and then explain it by using it.

***To solve $dy/dt = u(y)v(t)$, separate $dy/u(y)$ from $v(t)dt$ and integrate both sides***:

$$\int dy/u(y) = \int v(t)dt + C. \tag{6}$$

***Then substitute the initial condition to determine $C$, and solve for $y(t)$.***

**EXAMPLE 1** $dy/dt = y^2$ separates into $dy/y^2 = dt$. Integrate to reach $-1/y = t + C$. Substitute $t = 0$ and $y = y_0$ to find $C = -1/y_0$. Now solve for $y$:

$$-\frac{1}{y} = t - \frac{1}{y_0} \quad \text{and} \quad y = \frac{y_0}{1 - ty_0}.$$

This solution blows up (Figure 6.15a) when $t$ reaches $1/y_0$. If the bank pays interest on your deposit *squared* ($y' = y^2$), you soon have all the money in the world.

**EXAMPLE 2** $dy/dt = ty$ separates into $dy/y = t\,dt$. Then by integration $\ln y = \frac{1}{2}t^2 + C$. Substitute $t = 0$ and $y = y_0$ to find $C = \ln y_0$. The exponential of $\frac{1}{2}t^2 + \ln y_0$ gives $y = y_0 e^{t^2/2}$. When the interest rate is $c = t$, the exponent is $t^2/2$.

**EXAMPLE 3** $dy/dt = y + t$ is *not separable.* Method 1 survives by assuming $y =$

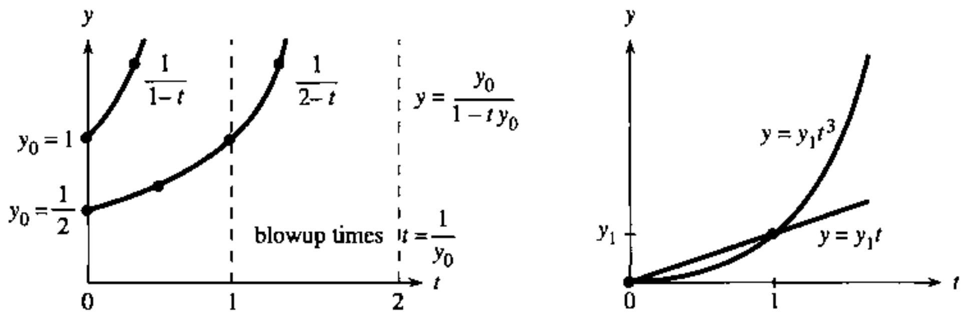



**Fig. 6.15** The solutions to separable equations $\frac{dy}{dt} = y^2$ and $\frac{dy}{dt} = n\frac{y}{t}$ or $\frac{dy}{y} = n\frac{dt}{t}$.

$Ae^t + B + Dt$—with an extra coefficient $D$ in Problem 23. Method 2 also succeeds—but not the separation method.

**EXAMPLE 4** Separate $dy/dt = ny/t$ into $dy/y = n\,dt/t$. By integration $\ln y = n \ln t + C$. Substituting $t = 0$ produces ln 0 and disaster. This equation cannot start from time zero (it divides by $t$). However $y$ can start from $y_1$ at $t = 1$, which gives $C = \ln y_1$. ***The solution is a power function*** $y = y_1 t^n$.

This was the first differential equation in the book (Section 2.2). The ratio of $dy/y$ to $dt/t$ is the "*elasticity*" in economics. These relative changes have units like dollars/dollars—they are dimensionless, and $y = t^n$ has constant elasticity $n$.

On log–log paper the graph of $\ln y = n \ln t + C$ is a *straight line with slope n*.

## THE LOGISTIC EQUATION

The simplest model of population growth is $dy/dt = cy$. The growth rate $c$ is the birth rate minus the death rate. If $c$ is constant the growth goes on forever—beyond the point where the model is reasonable. A population can't grow all the way to infinity! Eventually there is competition for food and space, and $y = e^{ct}$ must slow down.

***The true rate c depends on the population size*** $y$. It is a function $c(y)$ not a constant. The choice of the model is at least half the problem:

***Problem in biology or ecology***: Discover $c(y)$.

***Problem in mathematics***: Solve $dy/dt = c(y)y$.

Every model looks linear over a small range of $y$'s—but not forever. When the rate drops off, two models are of the greatest importance. The *Michaelis–Menten* equation has $c(y) = c/(y + K)$. The *logistic* equation has $c(y) = c - by$. It comes first.

The nonlinear effect is from "interaction." For two populations of size $y$ and $z$, the number of interactions is proportional to $y$ times $z$. ***The Law of Mass Action produces a quadratic term*** $byz$. It is the basic model for interactions and competition. Here we have one population competing within itself, so $z$ is the same as $y$. This competition slows down the growth, because $-by^2$ goes into the equation.

The basic model of *growth versus competition* is known as the ***logistic equation***:

$$dy/dt = cy - by^2. \tag{7}$$

Normally $b$ is very small compared to $c$. The growth begins as usual (close to $e^{ct}$). The competition term $by^2$ is much smaller than $cy$, *until y itself gets large*. Then $by^2$

(with its minus sign) slows the growth down. The solution follows an S-curve that we can compute exactly.

What are the numbers $b$ and $c$ for human population? Ecologists estimate the natural growth rate as $c = .029$/year. That is not the actual rate, because of $b$. About 1930, the world population was 3 billion. The $cy$ term predicts a yearly increase of (.029)(3 billion) = 87 million. The actual growth was more like $dy/dt = 60$ million/year. That difference of 27 million/year was $by^2$:

$$27 \text{ million/year} = b(3 \text{ billion})^2 \text{ leads to } b = 3 \cdot 10^{-12}/\text{year}.$$

Certainly $b$ is a small number (three trillionths) but its effect is not small. It reduces 87 to 60. What is fascinating is to calculate the ***steady state***, when the new term $by^2$ equals the old term $cy$. When these terms cancel each other, $dy/dt = cy - by^2$ is zero. The loss from competition balances the gain from new growth: $cy = by^2$ and $y = c/b$. The growth stops at this equilibrium point—the top of the S-curve:

$$y_\infty = \frac{c}{b} = \frac{.029}{3} 10^{12} \approx 10 \text{ billion people.}$$

According to Verhulst's logistic equation, *the world population is converging to* 10 *billion*. That is from the model. From present indications we are growing much faster. We will very probably go beyond 10 billion. The United Nations report in Section 3.3 predicts 11 billion to 14 billion.

***Notice a special point halfway to*** $y_\infty = c/b$. (In the model this point is at 5 billion.) It is the *inflection point* where the S-curve begins to bend down. The second derivative $d^2y/dt^2$ is zero. The slope $dy/dt$ is a maximum. It is easier to find this point from the differential equation (which gives $dy/dt$) than from $y$. Take one more derivative:

$$y'' = (cy - by^2)' = cy' - 2byy' = (c - 2by)y'. \tag{8}$$

The factor $c - 2by$ is zero at the inflection point $y = c/2b$, halfway up the S-curve.

## THE S-CURVE

***The logistic equation is solved by separating variables*** $y$ ***and*** $t$:

$$dy/dt = cy - by^2 \text{ becomes } \int dy/(cy - by^2) = \int dt. \tag{9}$$

The first question is whether we recognize this $y$-integral. *No*. The second question is whether it is listed in the cover of the book. *No*. The nearest is $\int dx/(a^2 - x^2)$, which can be reached with considerable manipulation (Problem 21). The third question is whether a general method is available. *Yes*. "Partial fractions" is perfectly suited to $1/(cy - by^2)$, and Section 7.4 gives the following integral of equation (9):

$$\ln \frac{y}{c - by} = ct + C \qquad \text{and then} \qquad \ln \frac{y_0}{c - by_0} = C. \tag{10}$$

That constant $C$ makes the solution correct at $t = 0$. The logistic equation is integrated, but the solution can be improved. Take exponentials of both sides to remove the logarithms:

$$\frac{y}{c - by} = e^{ct} \frac{y_0}{c - by_0}. \tag{11}$$

This contains the same growth factor $e^{ct}$ as in linear equations. But the logistic equation is not linear—it is not $y$ that increases so fast. According to (11), it is $y/(c-by)$ that grows to infinity. This happens when $c-by$ approaches zero.

The growth stops at $y=c/b$. That is the final population of the world (10 billion?).

We still need a formula for $y$. *The perfect* S*-curve is the graph of* $y=1/(1+e^{-t})$. It equals 1 when $t=\infty$, it equals $\frac{1}{2}$ when $t=0$, it equals 0 when $t=-\infty$. It satisfies $y'=y-y^2$, with $c=b=1$. The general formula cannot be so beautiful, because it allows any $c$, $b$, and $y_0$. To find the S-curve, multiply equation (11) by $c-by$ and solve for $y$:

$$y=\frac{c}{b+e^{-ct}(c-by_0)/y_0} \qquad \text{or} \qquad y=\frac{c}{b+de^{-ct}}. \tag{12}$$

When $t$ approaches infinity, $e^{-ct}$ approaches zero. The complicated part of the formula disappears. Then $y$ approaches its steady state $c/b$, the asymptote in Figure 6.16. The S-shape comes from the inflection point halfway up.

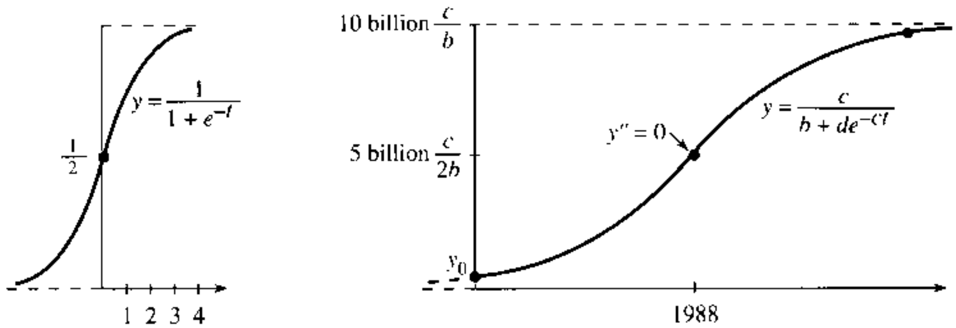



**Fig. 6.16** The standard S-curve $y=1/(1+e^{-t})$. The population S-curve (with prediction).

***Surprising observation***: $z=1/y$ *satisfies a linear equation*. By calculus $z'=-y'/y^2$. So

$$z'=\frac{-cy+by^2}{y^2}=-\frac{c}{y}+b=-cz+b. \tag{13}$$

This equation $z'=-cz+b$ is solved by an exponential $e^{-ct}$ plus a constant:

$$z=Ae^{-ct}+\frac{b}{c}=\left(\frac{1}{y_0}-\frac{b}{c}\right)e^{-ct}+\frac{b}{c}. \tag{14}$$

Turned upside down, $y=1/z$ is the S-curve (12). As $z$ approaches $b/c$, the S-curve approaches $c/b$. Notice that $z$ starts at $1/y_0$.

**EXAMPLE 1 (United States population)** The table shows the actual population and the model. Pearl and Reed used census figures for 1790, 1850, and 1910 to compute $c$ and $b$. In between, the fit is good but not fantastic. One reason is war—another is depression. Probably more important is immigration.† In fact the Pearl-Reed steady state $c/b$ is below 200 million, which the US has already passed. Certainly their model can be and has been improved. ***The*** 1990 ***census predicted a stop before*** 300 ***million***. For constant immigration $s$ we could still solve $y'=cy-by^2+s$ by partial fractions—but in practice the computer has taken over. The table comes from Braun's book *Differential Equations* (Springer 1975).

| Year | US Population | | Model |
|---|---|---|---|
| 1790 | 3.9 | = | 3.9 |
| 1800 | 5.3 | | 5.3 |
| 1810 | 7.2 | | 7.2 |
| 1820 | 9.6 | | 9.8 |
| 1830 | 12.9 | | 13.1 |
| 1840 | 17.1 | | 17.5 |
| 1850 | 23.2 | = | 23.2 |
| 1860 | 31.4 | | 30.4 |
| 1870 | 38.6 | | 39.4 |
| 1880 | 50.2 | | 50.2 |
| 1890 | 62.9 | | 62.8 |
| 1900 | 76.0 | | 76.9 |
| 1910 | 92.0 | = | 92.0 |
| 1920 | 105.7 | | 107.6 |
| 1930 | 122.8 | | 123.1 |
| 1940 | 131.7 | ≠ | 136.7 |
| 1950 | 150.7 | | 149.1 |

†Immigration does not enter for the world population model (at least not yet).

***Remark*** For good science the $y^2$ term should be explained and justified. It gave a nonlinear model that could be completely solved, but simplicity is not necessarily truth. The basic justification is this: In a population of size $y$, *the number of encounters is proportional to* $y^2$. If those encounters are fights, the term is $-by^2$. If those encounters *increase* the population, as some like to think, the sign is changed. There is a cooperation term $+by^2$, and the population increases very fast.

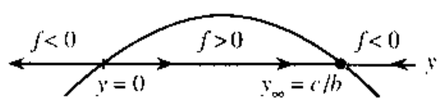
**EXAMPLE 5** $y' = cy + by^2$: *y goes to infinity in a finite time.*

**EXAMPLE 6** $y' = -dy + by^2$: *y dies to zero if* $y_0 < d/b$.

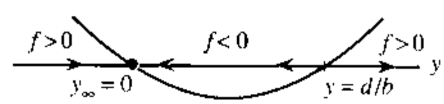
In Example 6 death wins. A small population dies out before the cooperation $by^2$ can save it. A population below $d/b$ is an endangered species.

The logistic equation can't predict oscillations—those go beyond $dy/dt = f(y)$.

***The y line*** Here is a way to understand every nonlinear equation $y' = f(y)$. Draw a "*y line*." Add arrows to show the sign of $f(y)$. When $y' = f(y)$ is positive, $y$ is increasing (***it follows the arrow to the right***). When $f$ is negative, $y$ goes to the left. When $f$ is zero, the equation is $y' = 0$ and $y$ is stationary:



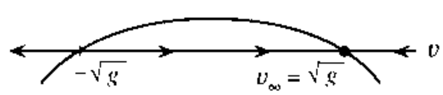

$y' = cy - by^2$ (this is $f(y)$)



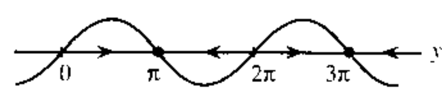

$y' = -dy + by^2$ (this is $f(y)$)

The arrows take you left or right, to the steady state or to infinity. Arrows go *toward* stable steady states. The arrows go *away*, when the stationary point is unstable. The ***y line*** shows which way $y$ moves and where it stops.

The terminal velocity of a falling body is $v_\infty = \sqrt{g}$ in Problem 6.7.54. For $f(y) = \sin y$ there are several steady states:




*falling body*: $dv/dt = g - v^2$




$dy/dt = \sin y$

**EXAMPLE 7** Kinetics of a chemical reaction $mA + nB \to pC$.

The reaction combines $m$ molecules of $A$ with $n$ molecules of $B$ to produce $p$ molecules of $C$. The numbers $m, n, p$ are 1, 1, 2 for hydrogen chloride: $H_2 + Cl_2 = 2\,HCl$. The ***Law of Mass Action*** says that the reaction rate is proportional to the product of the concentrations $[A]$ and $[B]$. Then $[A]$ decays as $[C]$ grows:

$$d[A]/dt = -r[A][B] \qquad \text{and} \qquad d[C]/dt = +k[A][B]. \tag{15}$$

Chemistry measures $r$ and $k$. Mathematics solves for $[A]$ and $[C]$. Write $y$ for the concentration $[C]$, the number of molecules in a unit volume. Forming those $y$ molecules drops the concentration $[A]$ from $a_0$ to $a_0 - (m/p)y$. Similarly $[B]$ drops from $b_0$ to $b_0 - (n/p)y$. The mass action law (15) contains $y^2$:

$$\frac{dy}{dt} = k\left(a_0 - \frac{m}{p}y\right)\left(b_0 - \frac{n}{p}y\right). \tag{16}$$

This fits our nonlinear model (Problem 33–34). We now find this same mass action in biology. You recognize it whenever there is a product of two concentrations.

### THE MM EQUATION $dy/dt = -cy/(y+K)$

Biochemical reactions are the keys to life. They take place continually in every living organism. Their mathematical description is not easy! Engineering and physics go far with linear models, while biology is quickly nonlinear. It is true that $y' = cy$ is extremely effective in first-order kinetics (Section 6.3), but nature builds in a nonlinear regulator.

It is ***enzymes*** that speed up a reaction. Without them, your life would be in slow motion. Blood would take years to clot. Steaks would take decades to digest. Calculus would take centuries to learn. The whole system is awesomely beautiful—DNA tells amino acids how to combine into useful proteins, and we get enzymes and elephants and Isaac Newton.

Briefly, the enzyme enters the reaction and comes out again. It is the ***catalyst***. Its combination with the substrate is an unstable intermediate, which breaks up into a new product and the enzyme (which is ready to start over).

Here are examples of catalysts, some good and some bad.

1. The platinum in a catalytic converter reacts with pollutants from the car engine. (But platinum also reacts with lead—ten gallons of leaded gasoline and you can forget the platinum.)
2. Spray propellants (CFC's) catalyze the change from ozone ($O_3$) into ordinary oxygen ($O_2$). This wipes out the ozone layer—our shield in the atmosphere.
3. Milk becomes yoghurt and grape juice becomes wine.
4. Blood clotting needs a whole cascade of enzymes, amplifying the reaction at every step. In hemophilia—the "Czar's disease"—the enzyme called Factor VIII is missing. A small accident is disaster; the bleeding won't stop.
5. Adolph's Meat Tenderizer is a protein from papayas. It predigests the steak. Tbe same enzyme (chymopapain) is injected to soften herniated disks.
6. Yeast makes bread rise. Enzymes put the sour in sourdough.

Of course, it takes enzymes to make enzymes. The maternal egg contains the material for a cell, and also half of the DNA. The fertilized egg contains the full instructions.

We now look at the Michaelis–Menten (MM) equation, to describe these reactions. It is based on the ***Law of Mass Action***. An enzyme in concentration $z$ converts a substrate in concentration $y$ by $dy/dt = -byz$. The rate constant is $b$, and you see the product of "enzyme times substrate." A similar law governs the other reactions (some go backwards). The equations are nonlinear, with no exact solution. It is typical of applied mathematics (and nature) that a pattern can still be found.

What happens is that the enzyme concentration $z(t)$ quickly drops to $z_0K/(y+K)$. The *Michaelis constant* $K$ depends on the rates (like $b$) in the mass action laws. Later the enzyme reappears ($z_\infty = z_0$). But by then the first reaction is over. Its law of mass action is effectively

$$\frac{dy}{dt} = -byz = -\frac{cy}{y+K} \tag{17}$$

with $c = bz_0K$. This is the ***Michaelis–Menten equation***—basic to biochemistry.

The rate $dy/dt$ is all-important in biology. Look at the function $cy/(y+K)$:

$$\text{when } y \text{ is large, } dy/dt \approx -c \qquad \text{when } y \text{ is small, } dy/dt \approx -cy/K.$$

The start and the finish operate at different rates, depending whether $y$ dominates $K$ or $K$ dominates $y$. The fastest rate is $c$.

A biochemist solves the MM equation by separating variables:

$$\int \frac{y+K}{y}\,dy = -\int c\,dt \quad \text{gives} \quad y + K \ln y = -ct + C. \tag{18}$$

Set $t = 0$ as usual. Then $C = y_0 + K \ln y_0$. The exponentials of the two sides are

$$e^y y^K = e^{-ct} e^{y_0} y_0^K. \tag{19}$$

We don't have a simple formula for $y$. We are lucky to get this close. A computer can quickly graph $y(t)$—and we see the dynamics of enzymes.

Problems 27–32 follow up the Michaelis–Menten theory. In science, concentrations and rate constants come with units. In mathematics, variables can be made dimensionless and constants become 1. We solve $dY/dT = Y/(Y+1)$ and then switch back to $y, t, c, K$. This idea applies to other equations too.

Essential point: ***Most applications of calculus come through differential equations.*** That is the language of mathematics—with populations and chemicals and epidemics obeying the same equation. Running parallel to $dy/dt = cy$ are the difference equations that come next.

## 6.5 EXERCISES

**Read-through questions**

The equations $dy/dt = cy$ and $dy/dt = cy + s$ and $dy/dt = u(y)v(t)$ are called __a__ because we can separate $y$ from $t$. Integration of $\int dy/y = \int c\,dt$ gives __b__. Integration of $\int dy/(y + s/c) = \int c\,dt$ gives __c__. The equation $dy/dx = -x/y$ leads to __d__. Then $y^2 + x^2 =$ __e__ and the solution stays on a circle.

The logistic equation is $dy/dt =$ __f__. The new term $-by^2$ represents __g__ when $cy$ represents growth. Separation gives $\int dy/(cy - by^2) = \int dt$, and the $y$-integral is $1/c$ times ln __h__. Substituting $y_0$ at $t = 0$ and taking exponentials produces $y/(c - by) = e^{ct}($__i__$)$. As $t \to \infty$, $y$ approaches __j__. That is the steady state where $cy - by^2 =$ __k__. The graph of $y$ looks like an __l__, because it has an inflection point at $y =$ __m__.

In biology and chemistry, concentrations $y$ and $z$ react at a rate proportional to $y$ times __n__. This is the Law of __o__. In a model equation $dy/dt = c(y)y$, the rate $c$ depends on __p__. The MM equation is $dy/dt =$ __q__. Separating variables yields $\int$ __r__ $dy =$ __s__ $= -ct + C$.

**Separate, integrate, and solve equations 1–8.**

**1** $dy/dt = y + 5, \quad y_0 = 2$

**2** $dy/dt = 1/y, \quad y_0 = 1$

**3** $dy/dx = x/y^2, \quad y_0 = 1$

**4** $dy/dx = y^2 + 1, \quad y_0 = 0$

**5** $dy/dx = (y+1)/(x+1), \quad y_0 = 0$

**6** $dy/dx = \tan y \cos x, \quad y_0 = 1$

**7** $dy/dt = y \sin t, \ y_0 = 1$

**8** $dy/dt = e^{t-y}, \quad y_0 = e$

**9** Suppose the rate of growth is proportional to $\sqrt{y}$ instead of $y$. Solve $dy/dt = c\sqrt{y}$ starting from $y_0$.

**10** The equation $dy/dx = ny/x$ for constant elasticity is the same as $d(\ln y)/d(\ln x) =$ ________. The solution is $\ln y =$ ________.

**11** When $c = 0$ in the logistic equation, the only term is $y' = -by^2$. What is the steady state $y_\infty$? How long until $y$ drops from $y_0$ to $\frac{1}{2}y_0$?

**12** Reversing signs in Problem 11, suppose $y' = +by^2$. At what time does the population explode to $y = \infty$, starting from $y_0 = 2$ (*Adam + Eve*)?

**Problems 13–26 deal with logistic equations $y' = cy - by^2$.**

**13** Show that $y = 1/(1 + e^{-t})$ solves the equation $y' = y - y^2$. Draw the graph of $y$ from starting values $\frac{1}{2}$ and $\frac{1}{3}$.

**14** (a) What logistic equation is solved by $y = 2/(1 + e^{-t})$?
(b) Find $c$ and $b$ in the equation solved by $y = 1/(1 + e^{-3t})$.

**15** Solve $z' = -z + 1$ with $z_0 = 2$. Turned upside down as in (13), what is $y = 1/z$?

16 By algebra find the S-curve (12) from $y = 1/z$ in (14).

17 How many years to grow from $y_0 = \frac{1}{2}c/b$ to $y = \frac{3}{4}c/b$? Use equation (10) for the time $t$ since the inflection point in 1988. When does $y$ reach 9 billion $= .9c/b$?

18 Show by differentiating $u = y/(c - by)$ that if $y' = cy - by^2$ then $u' = cu$. This explains the logistic solution (11) — it is $u = u_0 e^{ct}$.

19 Suppose Pittsburgh grows from $y_0 = 1$ million people in 1900 to $y = 3$ million in the year 2000. If the growth rate is $y' = 12{,}000$/year in 1900 and $y' = 30{,}000$/year in 2000, substitute in the logistic equation to find $c$ and $b$. What is the steady state? Extra credit: When does $y = y_\infty/2 = c/2b$?

20 Suppose $c = 1$ but $b = -1$, giving cooperation $y' = y + y^2$. Solve for $y(t)$ if $y_0 = 1$. When does $y$ become infinite?

21 Draw an S-curve through (0, 0) with horizontal asymptotes $y = -1$ and $y = 1$. Show that $y = (e^t - e^{-t})/(e^t + e^{-t})$ has those three properties. The graph of $y^2$ is shaped like ________.

22 To solve $y' = cy - by^3$ change to $u = 1/y^2$. Substitute for $y'$ in $u' = -2y'/y^3$ to find a linear equation for $u$. Solve it as in (14) but with $u_0 = 1/y_0^2$. Then $y = 1/\sqrt{u}$.

23 With $y = rY$ and $t = sT$, the equation $dy/dt = cy - by^2$ changes to $dY/dT = Y - Y^2$. Find $r$ and $s$.

24 In a change to $y = rY$ and $t = sT$, how are the initial values $y_0$ and $y_0'$ related to $Y_0$ and $Y_0'$?

25 A rumor spreads according to $y' = y(N - y)$. If $y$ people know, then $N - y$ don't know. The product $y(N - y)$ measures the number of meetings (to pass on the rumor).

(a) Solve $dy/dt = y(N - y)$ starting from $y_0 = 1$.

(b) At what time $T$ have $N/2$ people heard the rumor?

(c) This model is terrible because $T$ goes to ________ as $N \to \infty$. A better model is $y' = by(N - y)$.

26 Suppose $b$ and $c$ are both multiplied by 10. Does the middle of the S-curve get steeper or flatter?

**Problems 27–34 deal with mass action and the MM equation** $y' = -cy/(y + K)$.

27 Most drugs are eliminated acording to $y' = -cy$ but aspirin follows the MM equation. With $c = K = y_0 = 1$, does aspirin decay faster?

28 If you take aspirin at a constant rate $d$ (the maintenance dose), find the steady state level where $d = cy/(y + K)$. Then $y' = 0$.

29 Show that the rate $R = cy/(y + K)$ in the MM equation increases as $y$ increases, and find the maximum as $y \to \infty$.

30 Graph the rate $R$ as a function of $y$ for $K = 1$ and $K = 10$. (Take $c = 1$.) As the Michaelis constant increases, the rate ________. *At what value of y is* $R = \frac{1}{2}c$?

31 With $y = KY$ and $ct = KT$, find the "nondimensional" MM equation for $dY/dT$. From the solution $e^Y Y = e^{-T} e^{Y_0} Y_0$ recover the $y, t$ solution (19).

32 Graph $y(t)$ in (19) for different $c$ and $K$ (by computer).

33 The Law of Mass Action for $A + B \to C$ is $y' = k(a_0 - y)(b_0 - y)$. Suppose $y_0 = 0$, $a_0 = b_0 = 3$, $k = 1$. Solve for $y$ and find the time when $y = 2$.

34 In addition to the equation for $d[C]/dt$, the mass action law gives $d[A]/dt =$ ________.

35 Solve $y' = y + t$ from $y_0 = 0$ by assuming $y = Ae^t + B + Dt$. Find $A, B, D$.

36 Rewrite $cy - by^2$ as $a^2 - x^2$, with $x = \sqrt{b}y - c/2\sqrt{b}$ and $a =$ ________. Substitute for $a$ and $x$ in the integral taken from tables, to obtain the $y$-integral in the text:

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln \frac{a + x}{a - x} \qquad \int \frac{dy}{cy - by^2} = \frac{1}{c} \ln \frac{y}{c - by}$$

37 (Important) Draw the $y$-lines (with arrows as in the text) for $y' = y/(1 - y)$ and $y' = y - y^3$. Which steady states are approached from which initial values $y_0$?

38 Explain in your own words how the $y$-line works.

39 (a) Solve $y' = \tan y$ starting from $y_0 = \pi/6$ to find $\sin y = \frac{1}{2}e^t$.

(b) Explain why $t = 1$ is never reached.

(c) Draw arrows on the $y$-line to show that $y$ approaches $\pi/2$ — when does it get there?

40 Write the logistic equation as $y' = cy(1 - y/K)$. As $y'$ approaches zero, $y$ approaches ________. Find $y, y', y''$ at the inflection point.

## 6.6 Powers Instead of Exponentials

You may remember our first look at $e$. It is the special base for which $e^x$ has slope 1 at $x = 0$. That led to the great equation of exponential growth: *The derivative of* $e^x$ *equals* $e^x$. But our look at the actual number $e = 2.71828\ldots$ was very short.

It appeared as the limit of $(1 + 1/n)^n$. This seems an unnatural way to write down such an important number.

I want to show how $(1 + 1/n)^n$ and $(1 + x/n)^n$ arise naturally. They give ***discrete growth in finite steps***—with applications to compound interest. Loans and life insurance and money market funds use the discrete form of $y' = cy + s$. (We include extra information about bank rates, hoping this may be useful some day.) The applications in science and engineering are equally important. Scientific computing, like accounting, has *difference equations* in parallel with differential equations.

Knowing that this section will be full of formulas, I would like to jump ahead and tell you the best one. It is an infinite series for $e^x$. What makes the series beautiful is that ***its derivative is itself.***

Start with $y = 1 + x$. This has $y = 1$ and $y' = 1$ at $x = 0$. But $y''$ is zero, not one. Such a simple function doesn't stand a chance! No polynomial can be its own derivative, because the highest power $x^n$ drops down to $nx^{n-1}$. The only way is *to have no highest power*. We are forced to consider infinitely many terms—***a power series***—to achieve "derivative equals function."

To produce the derivative $1 + x$, we need $1 + x + \frac{1}{2}x^2$. Then $\frac{1}{2}x^2$ is the derivative of $\frac{1}{6}x^3$, which is the derivative of $\frac{1}{24}x^4$. The best way is to write the whole series at once:

$$\textbf{\textit{Infinite series}} \quad e^x = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \tfrac{1}{24}x^4 + \cdots. \tag{1}$$

This must be the greatest power series ever discovered. Its derivative is itself:

$$de^x/dx = 0 + 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \cdots = e^x. \tag{2}$$

The derivative of each term is the term before it. The integral of each term is the one after it (so $\int e^x dx = e^x + C$). The approximation $e^x \approx 1 + x$ appears in the first two terms. Other properties like $(e^x)(e^x) = e^{2x}$ are not so obvious. (Multiplying series is hard but interesting.) *It is not even clear why the sum is* 2.718... *when* $x = 1$. Somehow $1 + 1 + \frac{1}{2} + \frac{1}{6} + \cdots$ equals $e$. That is where $(1 + 1/n)^n$ will come in.

Notice that $x^n$ is divided by the product $1 \cdot 2 \cdot 3 \cdots n$. This is "*n factorial.*" Thus $x^4$ *is divided by* $1 \cdot 2 \cdot 3 \cdot 4 = 4! = 24$, and $x^5$ is divided by $5! = 120$. The derivative of $x^5/120$ is $x^4/24$, because 5 from the derivative cancels 5 from the factorial. In general $x^n/n!$ has derivative $x^{n-1}/(n-1)!$ Surprisingly $0!$ is 1.

Chapter 10 emphasizes that $x^n/n!$ becomes extremely small as $n$ increases. The infinite series adds up to a finite number—which is $e^x$. We turn now to discrete growth, which produces the same series in the limit.

This headline was on page one of the New York Times for May 27, 1990.

**213 Years After Loan, Uncle Sam is Dunned**

San Antonio, May 26—More than 200 years ago, a wealthy Pennsylvania merchant named Jacob DeHaven lent \$450,000 to the Continental Congress to rescue the troops at Valley Forge. That loan was apparently never repaid.

So Mr. DeHaven's descendants are taking the United States Government to court to collect what they believe they are owed. The total: \$141 billion if the interest is compounded daily at 6 percent, the going rate at the time. If compounded yearly, the bill is only \$98 billion.

The thousands of family members scattered around the country say they are not being greedy. "It's not the money—it's the principle of the thing," said Carolyn Cokerham, a DeHaven on her father's side who lives in San Antonio.

"You have to wonder whether there would even be a United States if this man had not made the sacrifice that he did. He gave everything he had."

The descendants say that they are willing to be flexible about the amount of settlement. But they also note that interest is accumulating at \$190 a second.

"None of these people have any intention of bankrupting the Government," said Jo Beth Kloecker, a lawyer from Stafford, Texas. Fresh out of law school, Ms. Kloecker accepted the case for less than the customary 30 percent contingency.

It is unclear how many descendants there are. Ms. Kloecker estimates that based on 10 generations with four children in each generation, there could be as many as half a million.

The initial suit was dismissed on the ground that the statute of limitations is six years for a suit against the Federal Government. The family's appeal asserts that this violates Article 6 of the Constitution, which declares as valid all debts owed by the Government before the Constitution was adopted.

Mr. DeHaven died penniless in 1812. He had no children.

## COMPOUND INTEREST

The idea of compound interest can be applied right away. Suppose you invest \$1000 at a rate of 100% (hard to do). If this is the *annual rate*, the interest after a year is another \$1000. You receive \$2000 in all. But if the interest is *compounded* you receive more:

after six months: Interest of \$500 is reinvested to give \$1500

end of year: New interest of \$750 (50% of 1500) gives \$2250 total.

The bank multiplied twice by 1.5 (1000 to 1500 to 2250). Compounding *quarterly* multiplies *four times* by 1.25 (1 for principal, .25 for interest):

$$\text{after one quarter the total is } 1000 + (.25)(1000) = 1250$$

$$\text{after two quarters the total is } 1250 + (.25)(1250) = 1562.50$$

$$\text{after nine months the total is } 1562.50 + (.25)(1562.50) = 1953.12$$

$$\text{after a full year the total is } 1953.12 + (.25)(1953.12) = 2441.41$$

Each step multiplies by $1 + (1/n)$, to add one $n$th of a year's interest—still at 100%:

$$\text{quarterly conversion: } (1 + 1/4)^4 \times 1000 = 2441.41$$

$$\text{monthly conversion: } (1 + 1/12)^{12} \times 1000 = 2613.04$$

$$\text{daily conversion: } (1 + 1/365)^{365} \times 1000 = 2714.57.$$

Many banks use 360 days in a year, although computers have made that obsolete. Very few banks use minutes (525,600 per year). Nobody compounds every second ($n = 31{,}536{,}000$). But some banks offer ***continuous compounding***. This is the limiting case ($n \to \infty$) that produces $e$:

$$\left(1 + \frac{1}{n}\right)^n \times 1000 \textit{ approaches } e \times 1000 = 2718.28.$$

**1.** *Quick method for* $(1 + 1/n)^n$: ***Take its logarithm. Use*** $\ln(1 + x) \approx x$ ***with*** $x = \dfrac{1}{n}$:

$$\ln\left(1+\frac{1}{n}\right)^n = n\ln\left(1+\frac{1}{n}\right) \approx n\left(\frac{1}{n}\right) = 1. \tag{3}$$

As $1/n$ gets smaller, this approximation gets better. The limit is 1. Conclusion: $(1 + 1/n)^n$ approaches the number whose logarithm is 1. Sections 6.2 and 6.4 define the same number (which is $e$).

**2.** *Slow method for* $(1 + 1/n)^n$: ***Multiply out all the terms. Then let*** $n \to \infty$.

This is a brutal use of the binomial theorem. It involves nothing smart like logarithms, but the result is a fantastic new formula for $e$.

$$\text{Practice for } n = 3: \quad \left(1+\frac{1}{3}\right)^3 = 1 + 3\left(\frac{1}{3}\right) + \frac{3\cdot 2}{1\cdot 2}\left(\frac{1}{3}\right)^2 + \frac{3\cdot 2\cdot 1}{1\cdot 2\cdot 3}\left(\frac{1}{3}\right)^3.$$

*Binomial theorem for any positive integer n:*

$$\left(1+\frac{1}{n}\right)^n = 1 + n\left(\frac{1}{n}\right) + \frac{n(n-1)}{1\cdot 2}\left(\frac{1}{n}\right)^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\left(\frac{1}{n}\right)^3 + \cdots + \left(\frac{1}{n}\right)^n. \tag{4}$$

Each term in equation (4) approaches a limit as $n \to \infty$. Typical terms are

$$\frac{n(n-1)}{1\cdot 2}\left(\frac{1}{n}\right)^2 \to \frac{1}{1\cdot 2} \qquad \text{and} \qquad \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\left(\frac{1}{n}\right)^3 \to \frac{1}{1\cdot 2\cdot 3}.$$

Next comes $1/1\cdot 2\cdot 3\cdot 4$. The sum of all those limits in (4) is our new formula for $e$:

$$\lim\left(1+\frac{1}{n}\right)^n = 1 + 1 + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \frac{1}{1\cdot 2\cdot 3\cdot 4} + \cdots = e. \tag{5}$$

In summation notation this is $\Sigma_{k=0}^{\infty} 1/k! = e$. The factorials give fast convergence:

$$1 + 1 + .5 + .16667 + .04167 + .00833 + .00139 + .00020 + .00002 = 2.71828.$$

Those nine terms give an accuracy that was not reached by $n = 365$ compoundings. A limit is still involved (to add up the whole series). ***You never see e without a limit!*** It can be defined by derivatives or integrals or powers $(1 + 1/n)^n$ or by an infinite series. Something goes to zero or infinity, and care is required.

All terms in equation (4) are below (or equal to) the corresponding terms in (5). ***The power*** $(1 + 1/n)^n$ ***approaches e from below.*** There is a steady increase with $n$. Faster compounding yields more interest. Continuous compounding at 100% yields $e$, as each term in (4) moves up to its limit in (5).

***Remark*** Change $(1 + 1/n)^n$ to $(1 + x/n)^n$. Now the binomial theorem produces $e^x$:

$$\left(1+\frac{x}{n}\right)^n = 1 + n\left(\frac{x}{n}\right) + \frac{n(n-1)}{1\cdot 2}\left(\frac{x}{n}\right)^2 + \cdots \text{ approaches } 1 + x + \frac{x^2}{1\cdot 2} + \cdots. \tag{6}$$

Please recognize $e^x$ on the right side! It is the infinite power series in equation (1). The next term is $x^3/6$ ($x$ can be positive or negative). This is a final formula for $e^x$:

**6L The limit of $(1 + x/n)^n$ is $e^x$. At $x = 1$ we find $e$.**

The logarithm of that power is $n \ln(1 + x/n) \approx n(x/n) = x$. The power approaches $e^x$.

To summarize: The quick method proves $(1 + 1/n)^n \to e$ by logarithms. The slow method (multiplying out every term) led to the infinite series. Together they show the agreement of all our definitions of $e$.

## DIFFERENCE EQUATIONS VS. DIFFERENTIAL EQUATIONS

We have the chance to see an important part of applied mathematics. This is not a course on differential equations, and it cannot become a course on difference equations. But it is a course with a purpose—*we aim to use what we know.* Our main application of $e$ was to solve $y' = cy$ and $y' = cy + s$. Now we solve the corresponding difference equations.

Above all, the goal is to see the connections. *The purpose of mathematics is to understand and explain patterns.* The path from "discrete to continuous" is beautifully illustrated by these equations. Not every class will pursue them to the end, but I cannot fail to show the pattern in a ***difference equation***:

$$y(t+1) = ay(t). \tag{7}$$

*Each step multiplies by the same number $a$.* The starting value $y_0$ is followed by $ay_0$, $a^2y_0$, and $a^3y_0$. The solution at discrete times $t = 0, 1, 2, \ldots$ is $y(t) = a^ty_0$.

***This formula $a^ty_0$ replaces the continuous solution $e^{ct}y_0$ of the differential equation.***

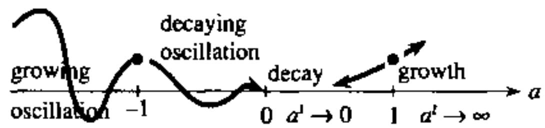



**Fig. 6.17** Growth for $|a| > 1$, decay for $|a| < 1$. Growth factor $a$ compares to $e^c$.

A source or sink (birth or death, deposit or withdrawal) is like $y' = cy + s$:

$$y(t+1) = ay(t) + s. \tag{8}$$

Each step multiplies by *$a$ and adds $s$.* The first outputs are

$$y(1) = ay_0 + s, \quad y(2) = a^2y_0 + as + s, \quad y(3) = a^3y_0 + a^2s + as + s.$$

We saw this pattern for differential equations—***every input $s$ becomes a new starting point.*** It is multiplied by powers of $a$. Since $s$ enters later than $y_0$, the powers stop at $t-1$. Algebra turns the sum into a clean formula by adding the geometric series:

$$y(t) = a^ty_0 + s\left[a^{t-1} + a^{t-2} + \cdots + a + 1\right] = a^ty_0 + s(a^t - 1)/(a-1). \tag{9}$$

**EXAMPLE 1** Interest at 8% from annual IRA deposits of $s = \$2000$ (here $y_0 = 0$).

The first deposit is at year $t = 1$. In a year it is multiplied by $a = 1.08$, because 8% is added. At the same time a new $s = 2000$ goes in. At $t = 3$ the first deposit has been multiplied by $(1.08)^2$, the second by 1.08, and there is another $s = 2000$. After year $t$,

$$y(t) = 2000(1.08^t - 1)/(1.08 - 1). \tag{10}$$

With $t = 1$ this is 2000. With $t = 2$ it is 2000 (1.08 + 1)—two deposits. Notice how $a - 1$ (the interest rate .08) appears in the denominator.

**EXAMPLE 2** Approach to steady state when $|a| < 1$. Compare with $c < 0$.

With $a > 1$, everything has been increasing. That corresponds to $c > 0$ in the differential equation (which is growth). But things die, and money is spent, so $a$ can be smaller than one. In that case $a^ty_0$ approaches zero—the starting balance disappears. What happens if there is also a source? *Every year half of the balance $y(t)$ is* *spent and a new \$2000 is deposited.* Now $a = \frac{1}{2}$:

$$y(t+1) = \tfrac{1}{2}y(t) + 2000 \quad \text{yields} \quad y(t) = (\tfrac{1}{2})^t y_0 + 2000\left[((\tfrac{1}{2})^t - 1)/(\tfrac{1}{2} - 1)\right].$$

The limit as $t \to \infty$ is an equilibrium point. As $(\frac{1}{2})^t$ goes to zero, $y(t)$ stabilizes to

$$y_\infty = 2000(0 - 1)/(\tfrac{1}{2} - 1) = 4000 = \textit{steady state}. \tag{11}$$

Why is 4000 steady? Because half is lost and the new 2000 makes it up again. ***The iteration is*** $y_{n+1} = \frac{1}{2}y_n + 2000$. ***Its fixed point is where*** $y_\infty = \frac{1}{2}y_\infty + 2000$.

In general the steady equation is $y_\infty = ay_\infty + s$. Solving for $y_\infty$ gives $s/(1-a)$. Compare with the steady differential equation $y' = cy + s = 0$:

$$y_\infty = -\frac{s}{c}\,\text{(differential equation)} \quad vs. \quad y_\infty = \frac{s}{1-a}\,\text{(difference equation)}. \tag{12}$$

**EXAMPLE 3** Demand equals supply when the price is right.

Difference equations are basic to economics. Decisions are made every year (by a farmer) or every day (by a bank) or every minute (by the stock market). There are three assumptions:

1. Supply next time depends on price this time: $S(t+1) = cP(t)$.
2. Demand next time depends on price next time: $D(t+1) = -dP(t+1) + b$.
3. Demand next time equals supply next time: $D(t+1) = S(t+1)$.

Comment on **3**: the price sets itself to make ***demand = supply***. The demand slope $-d$ is negative. The supply slope $c$ is positive. Those lines intersect at the competitive price, where supply equals demand. To find the difference equation, substitute **1** and **2** into **3**:

***Difference equation***: $-dP(t+1) + b = cP(t)$

***Steady state price***: $-dP_\infty + b = cP_\infty$. Thus $P_\infty = b/(c+d)$.

If the price starts above $P_\infty$, the difference equation brings it down. If below, the price goes up. When the price is $P_\infty$, it stays there. This is not news—economic theory depends on approach to a steady state. But convergence only occurs if $c < d$. *If supply is less sensitive than demand, the economy is stable.*

Blow-up example: $c = 2, b = d = 1$. The difference equation is $-P(t+1) + 1 = 2P(t)$. From $P(0) = 1$ the price oscillates as it grows: $P = -1, 3, -5, 11, \ldots$.

Stable example: $c = 1/2$, $b = d = 1$. The price moves from $P(0) = 1$ to $P(\infty) = 2/3$:

$$-P(t+1) + 1 = \frac{1}{2}P(t) \;\text{ yields }\; P = 1, \frac{1}{2}, \frac{3}{4}, \frac{5}{8}, \ldots, \text{ approaching } \frac{2}{3}.$$

Increasing $d$ gives greater stability. That is the effect of price supports. For $d = 0$ (fixed demand regardless of price) the economy is out of control.

## THE MATHEMATICS OF FINANCE

It would be a pleasure to make this supply-demand model more realistic—with curves, not straight lines. Stability depends on the slope—calculus enters. But we also have to be realistic about class time. I believe the most practical application is to solve ***the fundamental problems of finance***. Section 6.3 answered six questions about continuous interest. We now answer the same six questions when the annual rate is $x = .05 = 5\%$ and ***interest is compounded n times a year***.

First we compute ***effective rates***, higher than .05 because of compounding:

$$\text{compounded } \textit{quarterly} \left(1+\frac{.05}{4}\right)^4 = 1.0509 \quad \left[\textit{effective rate } .0509 = 5.09\%\right]$$

$$\text{compounded } \textit{continuously} \quad e^{.05} = 1.0513 \quad \left[\textit{effective rate } 5.13\%\right]$$

Now come the six questions. Next to the new answer (discrete) we write the old answer (continuous). One is algebra, the other is calculus. The time period is 20 years, so simple interest on $y_0$ would produce $(.05)(20)(y_0)$. That equals $y_0$ — money doubles in 20 years at 5% simple interest.

Questions **1** and **2** ask for the ***future value*** $y$ and ***present value*** $y_0$ *with compound interest n times a year*:

**1.** $y$ growing from $y_0$: $\quad y = \left(1+\frac{.05}{n}\right)^{20n} y_0 \qquad y = e^{(.05)(20)} y_0$

**2.** deposit $y_0$ to reach $y$: $\quad y_0 = \left(1+\frac{.05}{n}\right)^{-20n} y \qquad y_0 = e^{-(.05)(20)} y$

Each step multiplies by $a = (1 + .05/n)$. There are $20n$ steps in 20 years. Time goes backward in Question **2**. We divide by the growth factor instead of multiplying. The future value is greater than the present value (unless the interest rate is negative!). As $n \to \infty$ the discrete $y$ on the left approaches the continuous $y$ on the right.

Questions **3** and **4** connect $y$ to $s$ (with $y_0 = 0$ at the start). As soon as each $s$ is deposited, it starts growing. Then $y = s + as + a^2 s + \cdots$.

**3.** $y$ growing from deposits $s$: $\quad y = s\left[\frac{(1+.05/n)^{20n} - 1}{.05/n}\right] \qquad y = s\left[\frac{e^{(.05)(20)} - 1}{.05}\right]$

**4.** deposits $s$ to reach $y$: $\quad s = y\left[\frac{.05/n}{(1+.05/n)^{20n} - 1}\right] \qquad s = y\left[\frac{.05}{e^{(.05)(20)} - 1}\right]$

Questions **5** and **6** connect $y_0$ to $s$. This time $y$ is zero—***there is nothing left at the end.*** Everything is paid. The deposit $y_0$ is just enough to allow payments of $s$. This is an ***annuity***, where the bank earns interest on your $y_0$ while it pays you $s$ ($n$ times a year for 20 years). So your deposit in Question **5** is less than $20ns$.

Question **6** is the opposite—a ***loan***. At the start you borrow $y_0$ (instead of giving the bank $y_0$). You can earn interest on it as you pay it back. Therefore your payments have to total more than $y_0$. This is the calculation for car loans and mortgages.

**5.** ***Annuity***: Deposit $y_0$ to receive $20n$ payments of $s$:

$$y_0 = s\left[\frac{1 - (1+.05/n)^{-20n}}{.05/n}\right] \qquad y_0 = s\left[\frac{1 - e^{-(.05)(20)}}{.05}\right]$$

**6.** ***Loan***: Repay $y_0$ with $20n$ payments of $s$:

$$s = y_0\left[\frac{.05/n}{1 - (1+.05/n)^{-20n}}\right] \qquad s = y_0\left[\frac{.05}{1 - e^{-(.05)(20)}}\right]$$

Questions **2, 4, 6** are the inverses of **1, 3, 5**. Notice the pattern: There are three numbers $y$, $y_0$, and $s$. *One of them is zero each time.* If all three are present, go back to equation (9).

The algebra for these lines is in the exercises. *It is not calculus because $\Delta t$ is not $dt$.* All factors in brackets [ ] are listed in tables, and the banks keep copies. It might also be helpful to know their symbols. If a bank has interest rate $i$ per period over $N$ periods, then in our notation $a = 1 + i = 1 + .05/n$ and $t = N = 20n$:

$$\text{future value of } y_0 = \$1 \text{ (line 1): } y(N) = (1+i)^N$$

$$\text{present value of } y = \$1 \text{ (line 2): } y_0 = (1+i)^{-N}$$

$$\text{future value of } s = \$1 \text{ (line 3): } y(N) = s_{\overline{N}|i} = \left[(1+i)^N - 1\right]/i$$

$$\text{present value of } s = \$1 \text{ (line 5): } y_0 = a_{\overline{N}|i} = \left[1 - (1+i)^{-N}\right]/i$$

To tell the truth, I never knew the last two formulas until writing this book. The mortgage on my home has $N = (12)(25)$ monthly payments with interest rate $i = .07/12$. In 1972 tbe present value was \$42,000 = amount borrowed. I am now going to see if the bank is honest.†

***Remark*** In many loans, the bank computes interest on the amount paid back instead of the amount received. This is called ***discounting***. A loan of \$1000 at 5% for one year costs \$50 interest. Normally you receive \$1000 and pay back \$1050. *With discounting you receive* \$950 (called the proceeds) *and you pay back* \$1000. The true interest rate is higher than 5%—because the \$50 interest is paid on the smaller amount \$950. In this case the "discount rate" is $50/950 = 5.26\%$.

## SCIENTIFIC COMPUTING: DIFFERENTIAL EQUATIONS BY DIFFERENCE EQUATIONS

In biology and business, most events are discrete. In engineering and physics, time and space are continuous. Maybe at some quantum level it's all the same, but the equations of physics (starting with Newton's law $F = ma$) are differential equations. The great contribution of calculus is to model the rates of change we see in nature. But to *solve that model with a computer*, it needs to be made digital and discrete.

These paragraphs work with $dy/dt = cy$. It is the test equation that all analysts use, as soon as a new computing method is proposed. Its solution is $y = e^{ct}$, starting from $y_0 = 1$. Here we test Euler's method (nearly ancient, and not well thought of). He replaced $dy/dt$ by $\Delta y/\Delta t$:

$$\textbf{Euler's Method} \qquad \frac{y(t+\Delta t) - y(t)}{\Delta t} = cy(t). \tag{13}$$

The left side is $dy/dt$, in the limit $\Delta t \to 0$. We stop earlier, when $\Delta t > 0$.

The problem is to solve (13). Multiplying by $\Delta t$, the equation is

$$y(t+\Delta t) = (1 + c\Delta t)y(t) \qquad (\text{with } y(0) = 1).$$

Each step multiplies by $a = 1 + c\Delta t$, so $n$ steps multiply by $a^n$:

$$y = a^n = (1 + c\Delta t)^n \text{ at time } n\Delta t. \tag{14}$$

This is growth or decay, *depending on* $a$. The correct $e^{ct}$ is growth or decay, *depending on* $c$. ***The question is whether*** $a^n$ ***and*** $e^{ct}$ ***stay close***. Can one of them grow while the other decays? We expect the difference equation to copy $y' = cy$, but we might be wrong.

A good example is $y' = -y$. Then $c = -1$ and $y = e^{-t}$—the true solution decays.

---

†It's not. $s$ is too big. I knew it.

The calculator gives the following answers $a^n$ for $n = 2, 10, 20$:

| $\Delta t$ | $a = 1 + c\Delta t$ | $a^2$ | $a^{10}$ | $a^{20}$ |
|---|---|---|---|---|
| 3 | $-2$ | 4 | 1024 | 1048576 |
| 1 | 0 | 0 | 0 | 0 |
| 1/10 | .90 | .81 | .35 | .12 |
| 1/20 | .95 | .90 | .60 | .36 |

The big step $\Delta t = 3$ shows total instability (top row). The numbers blow up when they should decay. The row with $\Delta t = 1$ is equally useless (all zeros). In practice the magnitude of $c\Delta t$ must come down to .10 or .05. For accurate calculations it would have to be even smaller, unless we change to a better difference equation. That is the right thing to do.

Notice the two reasonable numbers. They are .35 and .36, approaching $e^{-1} = .37$. They come from $n = 10$ (with $\Delta t = 1/10$) and $n = 20$ (with $\Delta t = 1/20$). Those have the same clock time $n\Delta t = 1$:

$$\left(1 - \frac{1}{10}\right)^{10} = .35 \qquad \left(1 - \frac{1}{20}\right)^{20} = .36 \qquad \left(1 - \frac{1}{n}\right)^{n} \to e^{-1} = .37.$$

The main diagonal of the table is executing $(1 + x/n)^n \to e^x$ in the case $x = -1$.

Final question: ***How quickly are* .35 *and* .36 *converging to* $e^{-1} = .37$?** With $\Delta t = .10$ the error is .02. With $\Delta t = .05$ the error is .01. Cutting the time step in half cuts the error in half. We are not keeping enough digits to be sure, but *the error seems close to* $\frac{1}{5}\Delta t$. To test that, apply the "quick method" and estimate $a^n = (1 - \Delta t)^n$ from its logarithm:

$$\ln(1 - \Delta t)^n = n \ln(1 - \Delta t) \approx n\left[-\Delta t - \tfrac{1}{2}(\Delta t)^2\right] = -1 - \tfrac{1}{2}\Delta t. \tag{15}$$

The clock time is $n\Delta t = 1$. Now take exponentials of the far left and right:

$$a^n = (1 - \Delta t)^n \approx e^{-1}e^{-\Delta t/2} \approx e^{-1}(1 - \tfrac{1}{2}\Delta t). \tag{16}$$

The difference between $a^n$ and $e^{-1}$ is the last term $\frac{1}{2}\Delta t e^{-1}$. Everything comes down to one question: Is that error the same as $\frac{1}{5}\Delta t$? *The answer is yes*, because $e^{-1}/2$ is 1/5. If we keep only one digit, the prediction is perfect!

That took an hour to work out, and I hope it takes longer than $\Delta t$ to read. I wanted you to see *in use* the properties of $\ln x$ and $e^x$. The exact property $\ln a^n = n \ln a$ came first. In the middle of (15) was the key approximation $\ln(1 + x) \approx x - \frac{1}{2}x^2$, with $x = -\Delta t$. That $x^2$ term uses the second derivative (Section 6.4). At the very end came $e^x \approx 1 + x$.

A linear approximation shows convergence: $(1 + x/n)^n \to e^x$. A quadratic shows the error: proportional to $\Delta t = 1/n$. It is like using rectangles for areas, with error proportional to $\Delta x$. This minimal accuracy was enough to define the integral, and here it is enough to define $e$. It is completely unacceptable for scientific computing.

The trapezoidal rule, for integrals or for $y' = cy$, has errors of order $(\Delta x)^2$ and $(\Delta t)^2$. All good software goes further than that. Euler's first-order method could not predict the weather before it happens.

**Euler's Method for** $\dfrac{dy}{dt} = F(y, t)$: $\qquad \dfrac{y(t + \Delta t) - y(t)}{\Delta t} = F(y(t), t).$

## 6.6 EXERCISES

**Read-through questions**

The infinite series for $e^x$ is __a__. Its derivative is __b__. The denominator $n!$ is called "__c__" and it equals __d__. At $x = 1$ the series for $e$ is __e__.

To match the original definition of $e$, multiply out $(1 + 1/n)^n =$ __f__ (first three terms). As $n \to \infty$ those terms approach __g__ in agreement with $e$. The first three terms of $(1 + x/n)^n$ are __h__. As $n \to \infty$ they approach __i__ in agreement with $e^x$. Thus $(1 + x/n)^n$ approaches __j__. A quicker method computes $\ln(1 + x/n)^n \approx$ __k__ (first term only) and takes the exponential.

Compound interest ($n$ times in one year at annual rate $x$) multiplies by (__l__)$^n$. As $n \to \infty$, continuous compounding multiplies by __m__. At $x = 10\%$ with continuous compounding, \$1 grows to __n__ in a year.

The difference equation $y(t+1) = ay(t)$ yields $y(t) =$ __o__ times $y_0$. The equation $y(t+1) = ay(t) + s$ is solved by $y = a^t y_0 + s[1 + a + \cdots + a^{t-1}]$. The sum in brackets is __p__. When $a = 1.08$ and $y_0 = 0$, annual deposits of $s = 1$ produce $y =$ __q__ after $t$ years. If $a = \frac{1}{2}$ and $y_0 = 0$, annual deposits of $s = 6$ leave __r__ after $t$ years, approaching $y_\infty =$ __s__. The steady equation $y_\infty = ay_\infty + s$ gives $y_\infty =$ __t__.

When $i =$ interest rate per period, the value of $y_0 = \$1$ after $N$ periods is $y(N) =$ __u__. The deposit to produce $y(N) = 1$ is $y_0 =$ __v__. The value of $s = \$1$ deposited after each period grows to $y(N) =$ __w__. The deposit to reach $y(N) = 1$ is $s =$ __x__.

Euler's method replaces $y' = cy$ by $\Delta y = cy\Delta t$. Each step multiplies $y$ by __y__. Therefore $y$ at $t = 1$ is $(1 + c\Delta t)^{1/h} y_0$, which converges to __z__ as $\Delta t \to 0$. The error is proportional to __A__, which is too __B__ for scientific computing.

**1** Write down a power series $y = 1 - x + \cdots$ whose derivative is $-y$.

**2** Write down a power series $y = 1 + 2x + \cdots$ whose derivative is $2y$.

**3** Find *two* series that are equal to their second derivatives.

**4** By comparing $e = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \cdots$ with a larger series (whose sum is easier) show that $e < 3$.

**5** At 5% interest compute the output from \$1000 in a year with 6-month and 3-month and weekly compounding.

**6** With the quick method $\ln(1 + x) \approx x$, estimate $\ln(1 - 1/n)^n$ and $\ln(1 + 2/n)^n$. Then take exponentials to find the two limits.

**7** With the slow method multiply out the three terms of $(1 - \frac{1}{2})^2$ and the five terms of $(1 - \frac{1}{4})^4$. What are the first three terms of $(1 - 1/n)^n$, and what are their limits as $n \to \infty$?

**8** The slow method leads to $1 - 1 + 1/2! - 1/3! + \cdots$ for the limit of $(1 - 1/n)^n$. What is the sum of this infinite series — the exact sum and the sum after five terms?

**9** Knowing that $(1 + 1/n)^n \to e$, explain $(1 + 1/n)^{2n} \to e^2$ and $(1 + 2/N)^N \to e^2$.

**10** What are the limits of $(1 + 1/n^2)^n$ and $(1 + 1/n)^{n^2}$? OK to use a calculator to guess these limits.

**11** (a) The power $(1 + 1/n)^n$ (decreases) (increases) with $n$, as we compound more often. (b) The derivative of $f(x) = x\ln(1 + 1/x)$, which is ______, should be $(<0)(>0)$. This is confirmed by Problem 12.

**12** Show that $\ln(1 + 1/x) > 1/(x + 1)$ by drawing the graph of $1/t$. The area from $t = 1$ to $1 + 1/x$ is ______. The rectangle inside it has area ______.

**13** Take three steps of $y(t+1) = 2y(t)$ from $y_0 = 1$.

**14** Take three steps of $y(t+1) = 2y(t) + 1$ from $y_0 = 0$.

**Solve the difference equations 15–22.**

**15** $y(t+1) = 3y(t)$, $y_0 = 4$

**16** $y(t+1) = \frac{1}{2}y(t)$, $y_0 = 1$

**17** $y(t+1) = y(t) + 1$, $y_0 = 0$

**18** $y(t+1) = y(t) - 1$, $y_0 = 0$

**19** $y(t+1) = 3y(t) + 1$, $y_0 = 0$

**20** $y(t+1) = 3y(t) + s$, $y_0 = 1$

**21** $y(t+1) = ay(t) + s$, $y_0 = 0$

**22** $y(t+1) = ay(t) + s$, $y_0 = 5$

**In 23–26, which initial value produces $y_1 = y_0$ (steady state)?**

**23** $y(t+1) = 2y(t) - 6$

**24** $y(t+1) = \frac{1}{2}y(t) - 6$

**25** $y(t+1) = -y(t) + 6$

**26** $y(t+1) = -\frac{1}{2}y(t) + 6$

**27** In Problems 23 and 24, start from $y_0 = 2$ and take three steps to reach $y_3$. Is this approaching a steady state?

**28** For which numbers $a$ does $(1 - a^t)/(1 - a)$ approach a limit as $t \to \infty$ and what is the limit?

**29** The price $P$ is determined by supply = demand or $-dP(t+1) + b = cP(t)$. Which price $P$ is not changed from one year to the next?

**30** Find $P(t)$ from the supply-demand equation with $c = 1$, $d = 2$, $b = 8$, $P(0) = 0$. What is the steady state as $t \to \infty$?

**Assume 10% interest (so $a = 1 + i = 1.1$) in Problems 31–38.**

**31** At 10% interest compounded quarterly, what is the effective rate?

**32** At 10% interest compounded daily, what is the effective rate?

**33** Find the future value in 20 years of \$100 deposited now.

**34** Find the present value of \$1000 promised in twenty years.

35 For a mortgage of \$100,000 over 20 years, what is the monthly payment?

36 For a car loan of \$10,000 over 6 years, what is the monthly payment?

37 With annual compounding of deposits $s = \$1000$, what is the balance in 20 years?

38 If you repay $s = \$1000$ annually on a loan of \$8000, when are you paid up? (Remember interest.)

39 Every year two thirds of the available houses are sold, and 1000 new houses are built. What is the steady state of the housing market — how many are available?

40 If a loan shark charges 5% interest a month on the \$1000 you need for blackmail, and you pay \$60 a month, how much do you still owe after one month (and after a year)?

41 Euler charges $c = 100\%$ interest on his \$1 fee for discovering $e$. What do you owe (including the \$1) after a year with (a) no compounding; (b) compounding every week; (c) continuous compounding?

42 Approximate $(1 + 1/n)^n$ as in (15) and (16) to show that you owe Euler about $e - e/2n$. Compare Problem 6.2.5.

43 My Visa statement says monthly rate = 1.42% and yearly rate = 17%. What is the true yearly rate, since Visa compounds the interest? Give a formula or a number.

44 You borrow $y_0 = \$80,000$ at 9% to buy a house.

(a) What are your monthly payments $s$ over 30 years?

(b) How much do you pay altogether?

## 6.7 Hyperbolic Functions

This section combines $e^x$ with $e^{-x}$. Up to now those functions have gone separate ways—one increasing, the other decreasing. But two particular combinations have earned names of their own ($\cosh x$ and $\sinh x$):

$$\textbf{\textit{hyperbolic cosine}}\ \cosh x = \frac{e^x + e^{-x}}{2} \qquad\qquad \textbf{\textit{hyperbolic sine}}\ \sinh x = \frac{e^x - e^{-x}}{2}$$

The first name rhymes with "gosh". The second is usually pronounced "cinch".

The graphs in Figure 6.18 show that $\cosh x > \sinh x$. For large $x$ both hyperbolic functions come extremely close to $\frac{1}{2}e^x$. When $x$ is large and *negative*, it is $e^{-x}$ that dominates. Cosh $x$ still goes up to $+\infty$ while $\sinh x$ goes down to $-\infty$ (because $\sinh x$ has a minus sign in front of $e^{-x}$).

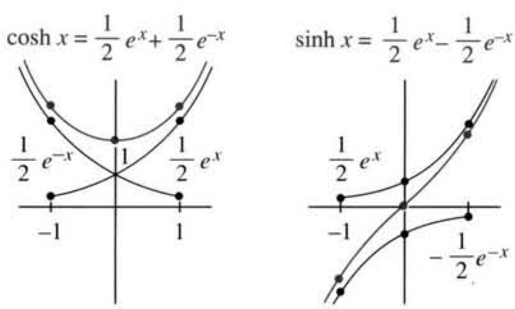



Fig. 6.18 Cosh $x$ and sinh $x$. The hyperbolic functions combine $\frac{1}{2}e^x$ and $\frac{1}{2}e^{-x}$.

Fig. 6.19 Gateway Arch courtesy of the St. Louis Visitors Commission.

The following facts come directly from $\frac{1}{2}(e^x + e^{-x})$ and $\frac{1}{2}(e^x - e^{-x})$:

$$\cosh(-x) = \cosh x \text{ and } \cosh 0 = 1 \quad (\text{cosh is } \textbf{\textit{even}} \text{ like the cosine})$$

$$\sinh(-x) = -\sinh x \text{ and } \sinh 0 = 0 \quad (\text{sinh is } \textbf{\textit{odd}} \text{ like the sine})$$

The graph of $\cosh x$ corresponds to a *hanging cable* (hanging under its weight). Turned upside down, it has the shape of the Gateway Arch in St. Louis. That must be the largest upside-down cosh function ever built. A cable is easier to construct than an arch, because gravity does the work. With the right axes in Problem 55, the height of the cable is a stretched-out cosh function called a ***catenary***:

$$y = a \cosh(x/a) \qquad \text{(cable tension/cable density} = a\text{)}.$$

Busch Stadium in St. Louis has 96 catenary curves, to match the Arch.

The properties of the hyperbolic functions come directly from the definitions. There are too many properties to memorize—and no reason to do it! One rule is the most important. ***Every fact about sines and cosines is reflected in a corresponding fact about*** $\sinh x$ ***and*** $\cosh x$. Often the only difference is a minus sign. Here are four properties:

**1.** $(\cosh x)^2 - (\sinh x)^2 = 1 \quad \left[\text{instead of } (\cos x)^2 + (\sin x)^2 = 1\right]$

$$\text{Check: } \left[\frac{e^x + e^{-x}}{2}\right]^2 - \left[\frac{e^x - e^{-x}}{2}\right]^2 = \frac{e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x}}{4} = 1$$

**2.** $\dfrac{d}{dx}(\cosh x) = \sinh x \quad \left[\text{instead of } \dfrac{d}{dx}(\cos x) = -\sin x\right]$

**3.** $\dfrac{d}{dx}(\sinh x) = \cosh x \quad \left[\text{like } \dfrac{d}{dx}\sin x = \cos x\right]$

**4.** $\displaystyle\int \sinh x\, dx = \cosh x + C \quad$ and $\quad \displaystyle\int \cosh x\, dx = \sinh x + C$

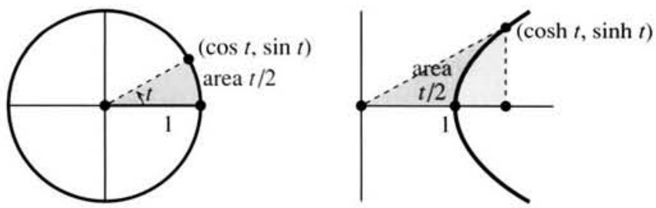



**Fig. 6.20** The unit circle $\cos^2 t + \sin^2 t = 1$ and the unit hyperbola $\cosh^2 t - \sinh^2 t = 1$.

Property 1 is the connection to hyperbolas. It is responsible for the "h" in cosh and sinh. Remember that $(\cos x)^2 + (\sin x)^2 = 1$ puts the point $(\cos x, \sin x)$ onto a *unit circle*. As $x$ varies, the point goes around the circle. The ordinary sine and cosine are "circular functions." Now look at $(\cosh x, \sinh x)$. Property 1 is $(\cosh x)^2 - (\sinh x)^2 = 1$, so this point travels on the *unit hyperbola* in Figure 6.20.

You will guess the definitions of the other four hyperbolic functions:

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \qquad \coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}} \qquad \operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

I think "tanh" is pronounceable, and "sech" is easy. The others are harder. Their properties come directly from $\cosh^2 x - \sinh^2 x = 1$. Divide by $\cosh^2 x$ and $\sinh^2 x$:

$$1 - \tanh^2 x = \operatorname{sech}^2 x \quad \text{and} \quad \coth^2 x - 1 = \operatorname{csch}^2 x$$

$$(\tanh x)' = \operatorname{sech}^2 x \quad \text{and} \quad (\operatorname{sech} x)' = -\operatorname{sech} x \tanh x$$

$$\int \tanh x \, dx = \int \frac{\sinh x}{\cosh x} dx = \ln(\cosh x) + C.$$

## INVERSE HYPERBOLIC FUNCTIONS

You remember the angles $\sin^{-1}x$ and $\tan^{-1}x$ and $\sec^{-1}x$. In Section 4.4 we differentiated those inverse functions by the chain rule. The main application was to *integrals*. If we happen to meet $\int dx/(1+x^2)$, it is $\tan^{-1}x + C$. The situation for $\sinh^{-1}x$ and $\tanh^{-1}x$ and $\operatorname{sech}^{-1}x$ is the same except for sign changes — which are expected for hyperbolic functions. We write down the ***three new derivatives***:

$$y = \sinh^{-1}x \text{ (meaning } x = \sinh y) \text{ has } \frac{dy}{dx} = \frac{1}{\sqrt{x^2+1}} \tag{1}$$

$$y = \tanh^{-1}x \text{ (meaning } x = \tanh y) \text{ has } \frac{dy}{dx} = \frac{1}{1-x^2} \tag{2}$$

$$y = \operatorname{sech}^{-1}x \text{ (meaning } x = \operatorname{sech} y) \text{ has } \frac{dy}{dx} = \frac{-1}{x\sqrt{1-x^2}} \tag{3}$$

Problems 44–46 compute $dy/dx$ from $1/(dx/dy)$. The alternative is to use logarithms. Since $\ln x$ is the inverse of $e^x$, we can express $\sinh^{-1}x$ and $\tanh^{-1}x$ and $\operatorname{sech}^{-1}x$ as logarithms. Here is $y = \tanh^{-1}x$:

$$y = \frac{1}{2}\ln\left[\frac{1+x}{1-x}\right] \text{ has slope } \frac{dy}{dx} = \frac{1}{2}\frac{1}{1+x} - \frac{1}{2}\frac{1}{1-x} = \frac{1}{1-x^2}. \tag{4}$$

The last step is an ordinary derivative of $\frac{1}{2}\ln(1+x) - \frac{1}{2}\ln(1-x)$. Nothing is new except the answer. But where did the logarithms come from? In the middle of the following identity, multiply above and below by $\cosh y$:

$$\frac{1+x}{1-x} = \frac{1+\tanh y}{1-\tanh y} = \frac{\cosh y + \sinh y}{\cosh y - \sinh y} = \frac{e^y}{e^{-y}} = e^{2y}.$$

Then $2y$ is the logarithm of the left side. This is the first equation in (4), and it is the third formula in the following list:

$$\sinh^{-1}x = \ln\left[x + \sqrt{x^2+1}\right] \qquad \cosh^{-1}x = \ln\left[x + \sqrt{x^2-1}\right]$$

$$\tanh^{-1}x = \frac{1}{2}\ln\left[\frac{1+x}{1-x}\right] \qquad \operatorname{sech}^{-1}x = \ln\left[\frac{1+\sqrt{1-x^2}}{x}\right]$$

***Remark 1*** Those are listed *only for reference*. If possible do not memorize them. The derivatives in equations (1), (2), (3) offer a choice of antiderivatives — either inverse functions or logarithms (most tables prefer logarithms). The inside cover of the book has

$$\int \frac{dx}{1-x^2} = \frac{1}{2}\ln\left[\frac{1+x}{1-x}\right] + C \quad \text{(in place of } \tanh^{-1}x + C).$$

***Remark 2*** Logarithms were not seen for $\sin^{-1}x$ and $\tan^{-1}x$ and $\sec^{-1}x$. You might wonder why. How does it happen that $\tanh^{-1}x$ is expressed by logarithms, when the parallel formula for $\tan^{-1}x$ was missing? Answer: *There must be a parallel formula.* To display it I have to reveal a secret that has been hidden throughout this section.

The secret is one of the great equations of mathematics. ***What formulas for*** $\cos x$ ***and*** $\sin x$ ***correspond to*** $\frac{1}{2}(e^x + e^{-x})$ ***and*** $\frac{1}{2}(e^x - e^{-x})$? With so many analogies (circular vs. hyperbolic) you would expect to find something. The formulas do exist, but ***they involve imaginary numbers.*** Fortunately they are very simple and there is no reason to withhold the truth any longer:

$$\cos x = \frac{1}{2}(e^{ix} + e^{-ix}) \quad \text{and} \quad \sin x = \frac{1}{2i}(e^{ix} - e^{-ix}). \tag{5}$$

It is the imaginary exponents that kept those identities hidden. Multiplying $\sin x$ by $i$ and adding to $\cos x$ gives Euler's unbelievably beautiful equation

$$\cos x + i \sin x = e^{ix}. \tag{6}$$

That is parallel to the non-beautiful hyperbolic equation $\cosh x + \sinh x = e^x$.

I have to say that (6) is infinitely more important than anything hyperbolic will ever be. The sine and cosine are far more useful than the sinh and cosh. So we end our record of the main properties, with exercises to bring out their applications.

## 6.7 EXERCISES

**Read-through questions**

Cosh $x$ = ___a___ and sinh $x$ = ___b___ and $\cosh^2 x - \sinh^2 x$ = ___c___. Their derivatives are ___d___ and ___e___ and ___f___. The point $(x, y) = (\cosh t, \sinh t)$ travels on the hyperbola ___g___. A cable hangs in the shape of a catenary $y$ = ___h___.

The inverse functions $\sinh^{-1}x$ and $\tanh^{-1}x$ are equal to $\ln[x + \sqrt{x^2+1}]$ and $\frac{1}{2}\ln$ ___i___. Their derivatives are ___j___ and ___k___. So we have two ways to write the anti___l___. The parallel to $\cosh x + \sinh x = e^x$ is Euler's formula ___m___. The formula $\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$ involves ___n___ exponents. The parallel formula for $\sin x$ is ___o___.

**1** Find $\cosh x + \sinh x$, $\cosh x - \sinh x$, and $\cosh x \sinh x$.

**2** From the definitions of $\cosh x$ and $\sinh x$, find their derivatives.

**3** Show that both functions satisfy $y'' = y$.

**4** By the quotient rule, verify $(\tanh x)' = \operatorname{sech}^2 x$.

**5** Derive $\cosh^2 x + \sinh^2 x = \quad \cosh 2x$, from the definitions.

**6** From the derivative of Problem 5 find $\sinh 2x$.

**7** The parallel to $(\cos x + i \sin x)^n = \cos nx + i \sin nx$ is a hyperbolic formula $(\cosh x + \sinh x)^n = \cosh nx +$ ________.

**8** Prove $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$ by changing to exponentials. Then the $x$-derivative gives $\cosh(x+y) =$ ________.

**Find the derivatives of the functions 9–18:**

**9** $\cosh(3x+1)$

**10** $\sinh x^2$

**11** $1/\cosh x$

**12** $\sinh(\ln x)$

**13** $\cosh^2 x + \sinh^2 x$

**14** $\cosh^2 x - \sinh^2 x$

**15** $\tanh \sqrt{x^2+1}$

**16** $(1 + \tanh x)/(1 - \tanh x)$

**17** $\sinh^6 x$

**18** $\ln(\operatorname{sech} x + \tanh x)$

**19** Find the minimum value of $\cosh(\ln x)$ for $x > 0$.

**20** From $\tanh x = \frac{3}{5}$ find sech $x$, cosh $x$, sinh $x$, coth $x$, csch $x$.

**21** Do the same if $\tanh x = -12/13$.

**22** Find the other five values if $\sinh x = 2$.

**23** Find the other five values if $\cosh x = 1$.

**24** Compute $\sinh(\ln 5)$ and $\tanh(2 \ln 4)$.

**Find antiderivatives for the functions in 25–32:**

**25** $\cosh(2x+1)$

**26** $x \cosh(x^2)$

**27** $\cosh^2 x \sinh x$

**28** $\tanh^2 x \operatorname{sech}^2 x$

**29** $\dfrac{\sinh x}{1 + \cosh x}$

**30** $\coth x = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

**31** $\sinh x + \cosh x$

**32** $(\sinh x + \cosh x)^n$

33 The triangle in Figure 6.20 has area $\frac{1}{2}\cosh t \sinh t$.

(a) Integrate to find the shaded area below the hyperbola

(b) For the area $A$ in red verify that $dA/dt = \frac{1}{2}$

(c) Conclude that $A = \frac{1}{2}t + C$ and show $C = 0$.

**Sketch graphs of the functions in 34–40.**

34 $y = \tanh x$ (with inflection point)

35 $y = \coth x$ (in the limit as $x \to \infty$)

36 $y = \operatorname{sech} x$

37 $y = \sinh^{-1} x$

38 $y = \cosh^{-1} x$ for $x \geqslant 1$

39 $y = \operatorname{sech}^{-1} x$ for $0 < x \leqslant 1$

40 $y = \tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ for $|x| < 1$

41 (a) Multiplying $x = \sinh y = \frac{1}{2}(e^y - e^{-y})$ by $2e^y$ gives $(e^y)^2 - 2x(e^y) - 1 = 0$. Solve as a quadratic equation for $e^y$.

(b) Take logarithms to find $y = \sinh^{-1} x$ and compare with the text.

42 (a) Multiplying $x = \cosh y = \frac{1}{2}(e^y + e^{-y})$ by $2e^y$ gives $(e^y)^2 - 2x(e^y) + 1 = 0$. Solve for $e^y$.

(b) Take logarithms to find $y = \cosh^{-1} x$ and compare with the text.

43 Turn (4) upside down to prove $y' = -1/(1-x^2)$, if $y = \coth^{-1} x$.

44 Compute $dy/dx = 1/\sqrt{x^2+1}$ by differentiating $x = \sinh y$ and using $\cosh^2 y - \sinh^2 y = 1$.

45 Compute $dy/dx = 1/(1-x^2)$ if $y = \tanh^{-1} x$ by differentiating $x = \tanh y$ and using $\operatorname{sech}^2 y + \tanh^2 y = 1$.

46 Compute $dy/dx = -1/x\sqrt{1-x^2}$ for $y = \operatorname{sech}^{-1} x$, by differentiating $x = \operatorname{sech} y$.

**From formulas (1), (2), (3) or otherwise, find antiderivatives in 47–52:**

47 $\int dx/(4-x^2)$

48 $\int dx/(a^2-x^2)$

49 $\int dx/\sqrt{x^2+1}$

50 $\int x\,dx/\sqrt{x^2+1}$

51 $\int dx/x\sqrt{1-x^2}$

52 $\int dx/\sqrt{1-x^2}$

53 Compute $\int_0^{1/2} \frac{dx}{1-x^2}$ and $\int_0^1 \frac{dx}{1-x^2}$.

54 A falling body with friction equal to velocity squared obeys $dv/dt = g - v^2$.

(a) Show that $v(t) = \sqrt{g}\tanh\sqrt{g}t$ satisfies the equation.

(b) Derive this $v$ yourself, by integrating $dv/(g - v^2) = dt$.

(c) Integrate $v(t)$ to find the distance $f(t)$.

55 A cable hanging under its own weight has slope $S = dy/dx$ that satisfies $dS/dx = c\sqrt{1+S^2}$. The constant $c$ is the ratio of cable density to tension.

(a) Show that $S = \sinh cx$ satisfies the equation.

(b) Integrate $dy/dx = \sinh cx$ to find the cable height $y(x)$, if $y(0) = 1/c$.

(c) Sketch the cable hanging between $x = -L$ and $x = L$ and find how far it sags down at $x = 0$.

56 The simplest nonlinear wave equation (Burgers' equation) yields a waveform $W(x)$ that satisfies $W'' = WW' - W'$. One integration gives $W' = \frac{1}{2}W^2 - W$.

(a) Separate variables and integrate: $dx = dW/(\frac{1}{2}W^2 - W) = -dW/(2-W) - dW/W$.

(b) Check $W' = \frac{1}{2}W^2 - W$.

57 A solitary water wave has a shape satisfying the *KdV* equation $y''' = y' - 6yy'$.

(a) Integrate once to find $y''$. Multiply the answer by $y'$.

(b) Integrate again to find $y'$ (all constants of integration are zero).

(c) Show that $y = \frac{1}{2}\operatorname{sech}^2(x/2)$ gives the shape of the "soliton."

58 Derive $\cos ix = \cosh x$ from equation (5). What is the cosine of the imaginary angle $i = \sqrt{-1}$?

59 Derive $\sin ix = i \sinh x$ from (5). What is $\sin i$?

60 The derivative of $e^{ix} = \cos x + i \sin x$ is ________.