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Probability: Random Isn't So Random
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Bayes' Rule, Independence

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Review of last class

- What does it mean if sets A , B , C are a partition of set D ?

Review of last class

- How do you calculate $P(A|B)$ using the formula for conditional probability?

Review of last class

- What is the difference between $P(A|B)$ and $P(B|A)$?

Review of last class

- If B causes A, what is $P(A|B)$?

Review of last class

- Does conditional probability require B to cause A?

Monty Hall Problem (from last class)

- Three doors: one has \$1 million behind it, the other two have nothing
- You pick one but it remains unclosed
- The host opens one door that reveals nothing (he knows which door has the prize)
- Before he opens your door (you only can pick one door), he gives you the choice of staying with your door or switching to the third door

Monty Hall Problem

Switch or Stay?

Bayes' Rule

- Finding $P(B|A)$ from $P(A|B)$
- Using the radar/airplane example from before
 - P = plane is there
 - R = radar registers a plane
 - $P(R|P) = P(\text{radar registers} \mid \text{plane is present})$
 - $P(P|R) = P(\text{plane is present} \mid \text{radar registers})$

Bayes' Rule

- How do we get $P(P|R)$?
- Draw the tree diagram again

Bayes' Rule

- $P(P|R)$ is just conditional probability


$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Bayes' Rule

- $P(P|R)$ is just conditional probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(P|R) = \frac{P(P \cap R)}{P(R)}$$

Bayes' Rule

- Finding $P(R)$
- The probability that the radar registers

$$P(P|R) = \frac{P(P \cap R)}{P(R)}$$

Bayes' Rule

- Finding $P(P \cap R)$
- The probability that the radar registers AND the plane is actually present

$$P(P|R) = \frac{P(P \cap R)}{P(R)}$$

Bayes' Rule

- Combine the probabilities
- $P(P|R) = ?$

Bayesian Probability

- Bayesian probability: how much you believe an event is happening based on evidence



Image courtesy of [the voyager](#) on flickr.

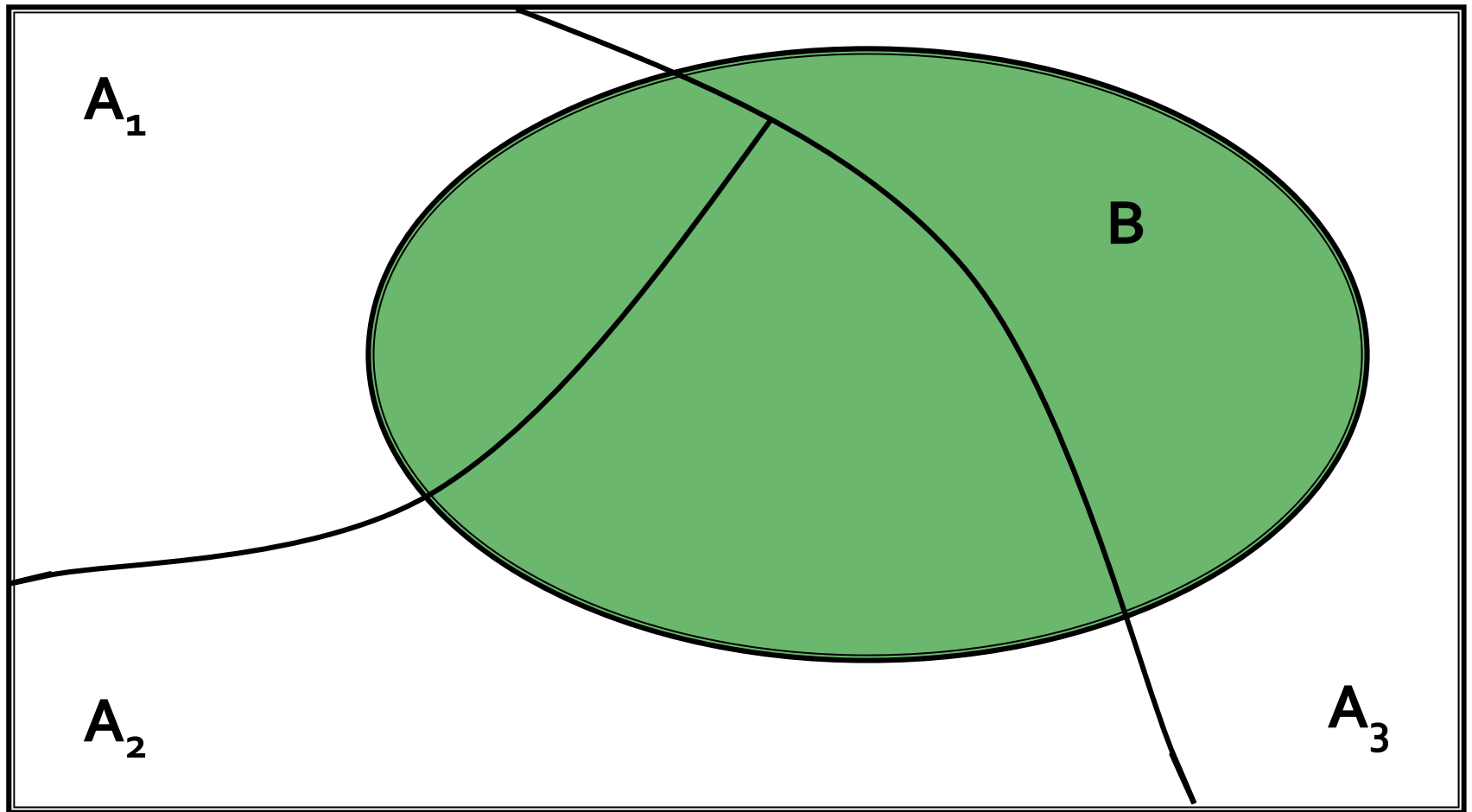
- The probability of a plane being present based on what a radar registers is more useful
- Example: military application

Assumptions for Bayes' Rule

Remember: we want to find $P(B|A)$ from $P(A|B)$

- The collection of all disjoint events $A_{1\dots n}$ are a partition of the entire sample space
- B is some event in the sample space

Assumptions for Bayes' Rule



Total Probability Theorem

- $P(B) = P(A_1 \cap B) + \dots + P(A_n \cap B)$

$$= P(A_1)P(B|A_1) + \dots + P(A_n)P(B|A_n)$$

[tree]

Equation for Bayes' Rule

- $P(A_i|B) = \frac{P(A_i)P(B|A_i)}{P(B)}$

$$= \frac{P(A_i)P(B|A_i)}{P(A_1)P(B|A_1) + \dots + P(A_n)P(B|A_n)}$$

Assumption for Bayes' Rule

- Sample space for the radar example?
- P = plane is present
- R = radar registers



Image courtesy of [davipt](#).



Image courtesy of [NASA](#).

Problem #1: Medical Diagnosis

- A test for a rare disease is assumed to be correct 95% of the time
- If the person has the disease, the test results are positive with probability 0.95
- If the person does not have the disease, the test results are negative with probability 0.95
- A random person drawn from a certain population has probability 0.001 of getting the disease
- Given that the person tested positive, what is the probability of having the disease?

Problem #1: Medical Diagnosis

- If you had to guess without any calculation, what would you think it was?
- $P(\text{has disease} \mid \text{tested positive})$?

Problem #1: Medical Diagnosis

- Sample space? (Draw a tree?)
- What are the probabilities?

What is Independence?

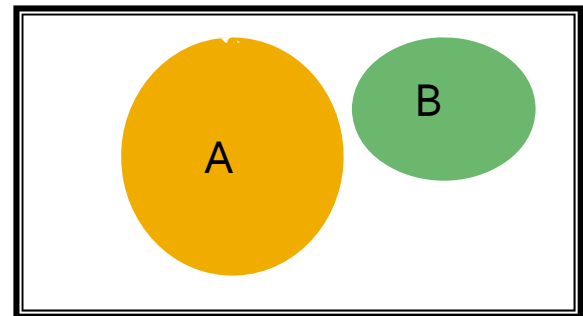
- What does it mean if $P(A|B) = P(A)$?

Deriving Test for Independence

- $P(A) = P(A|B)$
- Use definition of conditional probability
 - $P(A|B) = \frac{P(A \cap B)}{P(B)}$

Independent?

- Are two disjoint events independent?
- Remember the test for independence is
 $P(A \cap B) = P(A) P(B)$?



Example: Die Roll

- We throw a 4-sided die twice. Are the two successive rolls independent of each other?
- Sample space:
 $\{ (1,1) (1,2) (1,3) (1,4)$
 $(2,1) (2,2) (2,3) (2,4)$
 $(3,1) (3,2) (3,3) (3,4)$
 $(4,1) (4,2) (4,3) (4,4) \}$

Example: Die Roll

- Test for independence: $P(A)P(B) = P(A \cap B)$

Problem #2

Assume a 4-sided die again, rolled twice.

Are these events independent?

$A = \{1^{\text{st}} \text{ roll is } 1\}, \quad B = \{\text{sum of rolls is } 5\}$

Problem #3

Are these events independent?

$A = \{\text{maximum of the two rolls is } 2\}$

$B = \{\text{minimum of the two rolls is } 2\}$

Can you answer it intuitively? Mathematically?

Independence in real life

- Sometimes knowing whether events are independent is difficult
- In probability classes, problems will usually tell you to assume independence

Conditional Independence

- Definition:

$$P(A \cap B \mid C) = P(A \mid C)P(B \mid C)$$

given C, A and B are independent

- Another way to write this:

$$P(A \mid B \cap C) = P(A \mid C)$$

Example: Biased Coin Toss

- We have two coins: blue and red 
- We choose one of the coins at random (probability = $1/2$), and toss it twice
- Tosses are independent from each other given a coin
- The blue coin lands a head 99% of the time
- The red coin lands a head 1% of the time

Events: $H_1 = 1^{\text{st}}$ toss is a head
 $H_2 = 2^{\text{nd}}$ toss is a head

Example: Biased Coin Toss

- Tosses are independent from each other
GIVEN the choice of coin

← conditional independence

Problem #4: Biased Coin Toss

- What if you don't know what coin it is? Are the tosses still independent?

Summary

- Bayes' Rule
 - Total Probability Theorem
 - Independence
 - Conditional Independence
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- Things are not always what they seem! But with these tools you can calculate the probabilities accurately