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Summary: The Exponential $y = e^x$

Looking for a function $y(x)$ that equals its own derivative $\dfrac{dy}{dx}$

A differential equation! We start at $x = 0$ with $y = 1$

Infinite Series $y(x) = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots \left(\dfrac{x^n}{n!}\right) + \cdots$

Take derivative $\dfrac{dy}{dx} = 0 + 1 + x + \dfrac{x^2}{2!} + \cdots \left(\dfrac{x^{n-1}}{(n-1)!}\right) + \cdots$

Term by term $\dfrac{dy}{dx}$ agrees with y Limit step = add up this series

$n! = (n)(n-1)\cdots(1)$ grows much faster than x^n so terms get very small

At $x = 1$ the number $y(1)$ is called e. Set $x = 1$ in the series to find e

$e = 1 + 1 + \dfrac{1}{2} + \dfrac{1}{6} + \dfrac{1}{24} + \cdots = 2.71828...$

GOAL Show that $y(x)$ agrees with e^x for all x Series gives powers of e

Check that the series follows the rule $e^x e^X = e^{x+X}$ as in $e^2 e^3 = e^5$

Directly multiply series e^x times e^X to get e^{x+X}

$(1 + x + \frac{1}{2}x^2 + \cdots)$ times $(1 + X + \frac{1}{2}X^2 + \cdots)$ produces the right start

$1 + (x + X) + \frac{1}{2}(x + X)^2 + \cdots$ HIGHER TERMS ALSO WORK

Series gives us e^x for EVERY x, not just whole numbers

CHECK $\dfrac{de^x}{dx} = \lim \dfrac{e^{x+\Delta x} - e^x}{\Delta x} = e^x \left(\lim \dfrac{e^{\Delta x} - 1}{\Delta x}\right) = e^x$ YES!

SECOND KEY RULE $(e^x)^n = e^{nx}$ for every x and n

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Another approach to e^x uses multiplication instead of infinite sum

\$1 to start. Interest every day at yearly rate x

Multiply 365 times by $\left(1+\frac{x}{365}\right)$. End year with \$ $\left(1+\frac{x}{365}\right)^{365}$

Now pay n times in the year. End year with \$ $\left(1+\frac{x}{n}\right)^n \to \$\, e^x$ as $n \to \infty$

We are solving $\frac{\Delta y}{\Delta x} = y$ in n short steps $\Delta x = \frac{1}{n}$ Then limit as $\Delta x \to 0$

Practice Questions

1. What is the derivative of $\frac{x^{10}}{10!}$?

2. How to see that $\frac{x^n}{n!}$ gets small as $n \to \infty$?

Start with $\frac{x}{1}$ and $\frac{x^2}{2}$, possibly big. But we multiply by $\frac{x}{3}, \frac{x}{4}, \cdots$ which gets small.

3. Why is $\frac{1}{e^x}$ the same as e^{-x}?

4. Why is $e^{-1} = 1 - 1 + \frac{1}{2} - \frac{1}{6} + \cdots$ between $\frac{1}{3}$ and $\frac{1}{2}$? Then $2 < e < 3$.

5. Can you solve $\frac{dy}{dx} = y$ starting from $y = 3$ at $x = 0$?

Why is $y = 3e^x$ the right answer?

6. Can you solve $\frac{dy}{dx} = 5y$ starting from $y = 1$ at $x = 0$?

Why is $y = e^{5x}$ the right answer?

7. Why does $\frac{e^{\Delta x} - 1}{\Delta x}$ approach 1 as Δx gets smaller?