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Summary: Max-Min

To find minimum and maximum values of a function $y(x)$

Solve $\dfrac{dy}{dx} = 0$ to find points x^* where **slope = zero**

Test each x^* for a possible minimum or maximum

Example $\quad y(x) = x^3 - 12x \qquad \dfrac{dy}{dx} = 3x^2 - 12$

The slope is $\dfrac{dy}{dx} = 0$ at $x^* = 2$ and $x^* = -2$

At those points $y(2) = 8 - 24 = -16$ and $y(-2) = -8 + 24 = 16$

$x^* = 2$ is a minimum $\quad$ Look at $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = 2^{\text{nd}}$ derivative

$\dfrac{d^2y}{dx^2} =$ derivative of $3x^2 - 12$. 2^{nd} derivative is $6x$.

$\dfrac{d^2y}{dx^2} > 0 \quad \dfrac{dy}{dx}$ increases $\quad$ slope goes from down to up at x^*

The bending is upwards and this x^* is a **minimum**

$\dfrac{d^2y}{dx^2} < 0 \quad \dfrac{dy}{dx}$ decreases $\quad$ slope goes from up to down at x^*

The bending is downwards and x^* is a **maximum**

Find the maximum of $y(x) = \sin x + \cos x \qquad \dfrac{dy}{dx} = \cos x - \sin x$

The slope is zero when $\cos x = \sin x$ at $x^* = 45$ degrees $= \dfrac{\pi}{4}$ radians

That point x^* has $y = \sin\dfrac{\pi}{4} + \cos\dfrac{\pi}{4} = \dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{2}}{2} = \sqrt{2}$

The second derivative is $\dfrac{d^2y}{dx^2} = -\sin x - \cos x$

At $x^* = \dfrac{\pi}{4}$ this is $< 0 \quad y$ is bending down $\quad x^*$ is a maximum

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$\frac{d^2y}{dx^2} > 0$ the curve bends up and $\frac{d^2y}{dx^2} < 0$ the curve bends down

Bending changes at a **point of inflection** where $\frac{d^2y}{dx^2} = 0$

Which x^* gives minimum of $y = (x-1)^2 + (x-2)^2 + (x-6)^2$?

You can write $y = (x^2 - 2x + 1) + (x^2 - 4x + 4) + (x^2 - 12x + 36)$

The slope is $\frac{dy}{dx} = 2x - 2 + 2x - 4 + 2x - 12 = 0$ at minimum

Then $6x^* = 18$ and $x^* = 3$ This is the average of 1, 2, 6

Key step for max/min word problems is to choose meaning for x

Practice Questions

1. Which x^* gives the minimum of $y(x) = x^2 + 2x$? Solve $\frac{dy}{dx} = 0$.

2. Find $\frac{d^2y}{dx^2}$ for $y(x) = x^2 + 2x$.
This is > 0 so parabola bends up.

3. Find the maximum height of $y(x) = 2 + 6x - x^2$. Solve $\frac{dy}{dx} = 0$.

4. Find $\frac{d^2y}{dx^2}$ to show that this parabola bends down.

5. For $y(x) = x^4 - 2x^2$ show that $\frac{dy}{dx} = 0$ at $x = -1, 0, 1$.
Find $y(-1), y(0), y(-1)$.

6. Now $\frac{dy}{dx} = 4x^3 - 4x$. What is the second derivative $\frac{d^2y}{dx^2}$?

7. At a minimum point explain why $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$.

8. Bending down $\left(\frac{d^2y}{dx^2} < 0\right)$ changes to bending up $\left(\frac{d^2y}{dx^2} > 0\right)$ at a point of _____ : At this point $\frac{d^2y}{dx^2} = 0$
Does $y = x^2$ have such a point? Does $y = \sin x$ have such a point?

9. Suppose $x + X = 12$. What is the maximum of x times X?
This question asks for the maximum of $y = x(12 - x) = 12x - x^2$.
Find where the slope $\frac{dy}{dx} = 12 - 2x$ is zero. What is x times X?