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Summary: Big Picture - Integrals

Key problem Recover the integral $y(x)$ from its derivative $\frac{dy}{dx}$

Find total distance traveled from record of the speed

Find total height knowing the slope since the start

Simplest way Recognize $\frac{dy}{dx}$ as derivative of a known $y(x)$

If $\frac{dy}{dx} = x^3$ then its **integral** $y(x)$ (Function (1)) was $\frac{1}{4}x^4 + C$

If $\frac{dy}{dx} = e^x$ then $y = e^x + C$

If $\frac{dy}{dx} = e^{2x}$ then $y = \frac{1}{2}e^{2x} + C$

Integral Calculus is the reverse of Differential Calculus

$y(x) = \int \frac{dy}{dx}\, dx$ adds up the whole history of slopes $\frac{dy}{dx}$

Integral is like sum Derivative is like difference

Sums	y_0		y_1		y_2		y_3		y_4
Differences		$y_1 - y_0$		$y_2 - y_1$		$y_3 - y_2$		$y_4 - y_3$	

Notice cancellation $(y_1 - y_0) + (y_2 - y_1) = y_2 - y_0$ = change in height

Divide and multiply differences by step size Δx

Sum of $\frac{\Delta y}{\Delta x}\Delta x = \frac{y_1 - y_0}{\Delta x}\Delta x + \frac{y_2 - y_1}{\Delta x}\Delta x = y_2 - y_0$

$\Delta x \to 0$ Sum changes to integral $\int \frac{dy}{dx}\, dx = y_{\text{end}} - y_{\text{start}}$

Fundamental Theorem of Calculus $\int \frac{dy}{dx}\, dx = y(x) + C$

The integral reverses the derivative and brings back $y(x)$

Integration and Differentiation are inverse operations

Opposite order too $\frac{d}{dx}\int_0^x s(t)\, dt = s(x)$

KEY What is the meaning of an integral $\int_0^x s(t)\, dt$?

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Example Slope $s(t) = 6t$ (increasing speed)

Method 1 $y = 3t^2$ has the required derivative $6t$

Method 2 The triangle under the graph of $s(t)$ has area $3t^2$

From 0 to t, base $= t$ and height $= 6t$ and area $= \frac{1}{2}t(6t)$.

[Most shapes are more difficult! Area comes from integrating s]

Method 3 (fundamental) Short time steps each at constant speed

This **limiting step** defines the integral of $s(t)$

In a step Δt, distance is close to $s(t^*)\Delta t$

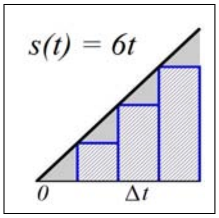
t^* = starting time for that step $s(t^*)$ = starting speed

Not exact because speed changes a little within time Δt

Total distance is close to sum of short distances $s(t^*)\Delta t$

Total distance becomes exact as $\Delta t \to 0$ and **sum** $\to$ **integral**

Picture of each step shows a tall thin rectangle



$s(t^*)\Delta t$ = height times base
= area of rectangle
t^* = start point of the time step

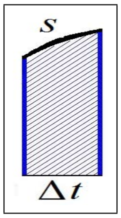
Sum of $s(t^*)\Delta t$ = total area of all rectangles

Now $\Delta t \to 0$ The rectangles fill up the triangle

Integral of $s(t)\,dt$ = exact area $y(t)$ under the graph

Fundamental Theorem Area $y(t)$ has desired derivative $s(t)$

Δy is the thin area under $s(t)$ between t and $t + \Delta t$



Δt is the base of that thin "rectangle"

$\frac{\Delta y}{\Delta t}$ is the height of that thin "rectangle"

This height approaches $s(t)$ as the base $\Delta t \to 0$

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Practice Questions

1. What functions $y(t)$ have the constant derivative $s(t) = 7$?
2. What is the area from 0 to t under the graph of $s(t) = 7$?
3. From $t = 0$ to 2, find the integral $\int_0^2 7\, dt =$ _____.
4. What function $y(t)$ has the derivative $s(t) = 7 + 6t$?
5. From $t = 0$ to 2, find area = integral $\int_0^2 (7 + 6t)\, dt$.
6. At this instant $t = 2$, what is $\dfrac{d(\text{area})}{dt}$?

7. From 0 to t, the area under the curve $s = e^t$ IS NOT $y = e^t$.
If t is small, the area must be small. But $t = 0$ has $y = e^0 = 1$.
8. From 0 to t, the correct area under $s = e^t$ is $y = e^t - 1$.
The slope $\dfrac{dy}{dt}$ is _____ and now $y(0) =$ _____
9. Notice y_0 in $(y_1 - y_0) + (y_2 - y_1) + (y_3 - y_2) =$ _____.
The sum of $\Delta y = \dfrac{\Delta y}{\Delta t} \Delta t$ becomes the integral of $\dfrac{dy}{dt} dt$
The area under $s(t)$ from 0 to t becomes $y(t) - y(0)$.