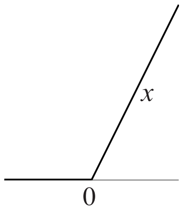
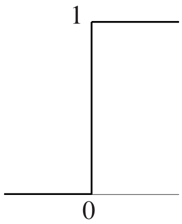
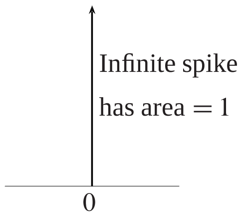
Summary: Six Functions, Six Rules, Six Theorems

Integrals	**Six Functions**	***Derivatives***
$x^{n+1}/(n+1),\ n \neq -1$	$\boldsymbol{x^n}$	nx^{n-1}
$-\cos x$	$\boldsymbol{\sin x}$	$\cos x$
$\sin x$	$\boldsymbol{\cos x}$	$-\sin x$
e^{cx}/c	$\boldsymbol{e^{cx}}$	ce^{cx}
$x \ln x - x$	$\boldsymbol{\ln x}$	$1/x$
Ramp function	**Step function**	**Delta function**







Six Rules of Differential Calculus

1. The derivative of $\boldsymbol{af(x)+bg(x)}$ is $a\dfrac{df}{dx}+b\dfrac{dg}{dx}$ **Sum**

2. The derivative of $\boldsymbol{f(x)g(x)}$ is $f(x)\dfrac{dg}{dx}+g(x)\dfrac{df}{dx}$ **Product**

3. The derivative of $\dfrac{\boldsymbol{f(x)}}{\boldsymbol{g(x)}}$ is $\left(g\dfrac{df}{dx}-f\dfrac{dg}{dx}\right)\Big/ g^2$ **Quotient**

4. The derivative of $\boldsymbol{f(g(x))}$ is $\dfrac{df}{dy}\dfrac{dy}{dx}$ where $y=g(x)$ **Chain**

5. The derivative of $x=\boldsymbol{f^{-1}(y)}$ is $\dfrac{dx}{dy}=\dfrac{1}{dy/dx}$ **Inverse**

6. When $f(x)\to 0$ and $g(x)\to 0$ as $x\to a$, what about $f(x)/g(x)$? **l'Hôpital**

$$\lim \frac{f(x)}{g(x)} = \lim \frac{df/dx}{dg/dx}$$ if these limits exist. Normally this is $\dfrac{\boldsymbol{f'(a)}}{\boldsymbol{g'(a)}}$

Fundamental Theorem of Calculus

If $f(x)=\displaystyle\int_a^x s(t)\,dt$ then **derivative of integral** $=\dfrac{df}{dx}=\boldsymbol{s(x)}$

If $\dfrac{df}{dx}=s(x)$ then **integral of derivative** $=\displaystyle\int_a^b s(x)\,dx=\boldsymbol{f(b)-f(a)}$

Both parts assume that $s(x)$ is a continuous function.

All Values Theorem Suppose $f(x)$ is a continuous function for $a \leqslant x \leqslant b$. Then on that interval, $f(x)$ reaches its maximum value M and its minimum m. And $f(x)$ takes all values between m and M (there are no jumps).

Mean Value Theorem If $f(x)$ has a derivative for $a \leqslant x \leqslant b$ then

$$\frac{f(b)-f(a)}{b-a} = \frac{df}{dx}(c) \quad \text{at some } c \text{ between } a \text{ and } b$$

"At some moment c, instant speed = average speed"

Taylor Series Match all the derivatives $f^{(n)} = d^n f/dx^n$ at the basepoint $x = a$

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \cdots$$

$$= \boldsymbol{\sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(a)\,(x-a)^n}$$

Stopping at $(x-a)^n$ leaves the error $f^{n+1}(c)(x-a)^{n+1}/(n+1)!$

[c is somewhere between a and x] [$n = 0$ is the Mean Value Theorem]

The Taylor series looks best around $a = 0$ $\quad f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0)\, x^n$

Binomial Theorem shows Pascal's triangle

$(1+x)$	$\mathbf{1} + \mathbf{1}x$
$(1+x)^2$	$\mathbf{1} + \mathbf{2}x + \mathbf{1}x^2$
$(1+x)^3$	$\mathbf{1} + \mathbf{3}x + \mathbf{3}x^2 + \mathbf{1}x^3$
$(1+x)^4$	$\mathbf{1} + \mathbf{4}x + \mathbf{6}x^2 + \mathbf{4}x^3 + \mathbf{1}x^4$

Those are just the Taylor series for $f(x) = (1+x)^p$ when $p = 1, 2, 3, 4$

$$\begin{array}{lllll} f^{(n)}(x) = & (1+x)^p & p(1+x)^{p-1} & p(p-1)(1+x)^{p-2} & \cdots \\ f^{(n)}(0) = & \mathbf{1} & \boldsymbol{p} & \boldsymbol{p(p-1)} & \cdots \end{array}$$

Divide by $n!$ to find the Taylor coefficients = **Binomial coefficients**

$$\frac{1}{n!} f^{(n)}(0) = \frac{p(p-1)\cdots(p-n+1)}{n(n-1)\cdots(1)} = \frac{p!}{(p-n)!\,n!} = \binom{\boldsymbol{p}}{\boldsymbol{n}}$$

The series stops at x^n when $p = n$ Infinite series for other p

Every $(1+x)^p = 1 + px + \frac{p(p-1)}{(2)(1)}x^2 + \frac{p(p-1)(p-2)}{(3)(2)(1)}x^3 + \cdots$

Practice Questions

1. Check that the derivative of $y = x\ln x - x$ is $dy/dx = \ln x$.

2. The "sign function" is $S(x) = \begin{cases} 1 & \text{for } x \geq 0 \\ -1 & \text{for } x < 0 \end{cases}$

   What ramp function $F(x)$ has $\dfrac{dF}{dx} = S(x)$? F is the integral of S.

   Why is the derivative $\dfrac{dS}{dx} = 2\,\textbf{delta}(x)$? (Infinite spike at $x = 0$ with area 2)

3. (l'Hôpital) What is the limit of $\dfrac{2x+3x^2}{5x+7x^2}$ as $x \to 0$? What about $x \to \infty$?

4. l'Hôpital's Rule says that $\lim_{x\to 0} \dfrac{f(x)}{x} = ??$ when $f(0) = 0$. Here $g(x) = x$.

5. **Derivative is like Difference Integral is like Sum**

   Difference of sums If $f_n = s_1 + s_2 + \cdots + s_n$, what is $f_n - f_{n-1}$?

   Sums of differences What is $(f_1 - f_0) + (f_2 - f_1) + \cdots + (f_n - f_{n-1})$?

   Those are the **Fundamental Theorems** of **"Difference Calculus"**

6. Draw a non-continuous graph for $0 \leq x \leq 1$ where your function does NOT reach its maximum value.

7. For $f(x) = x^2$, which in-between point c gives $\dfrac{f(5) - f(1)}{5-1} = \dfrac{df}{dx}(c)$?

8. If your average speed on the Mass Pike is 75, then at some instant your speedometer will read _____ .

9. Find three Taylor coefficients A, B, C for $\sqrt{1+x}$ (around $x = 0$).

$$(1+x)^{\frac{1}{2}} = A + Bx + Cx^2 + \cdots$$

10. Find the Taylor (= Binomial) series for $f = \dfrac{1}{1+x}$ around $x = 0$ $(p = -1)$.

Resource: Highlights of Calculus
Gilbert Strang