

Product and Quotient Rules

**Goal** To find the derivative of  $y = f(x)g(x)$  from  $\frac{df}{dx}$  and  $\frac{dg}{dx}$

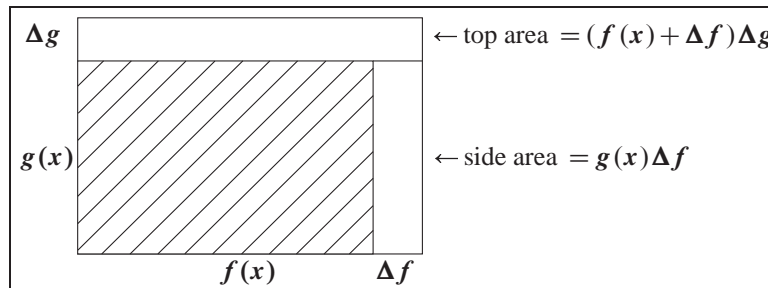
**Idea** Write  $\Delta y = f(x + \Delta x)g(x + \Delta x) - f(x)g(x)$  by separating  $\Delta f$  and  $\Delta g$

That same  $\Delta y$  is  $f(x + \Delta x)[g(x + \Delta x) - g(x)] + g(x)[f(x + \Delta x) - f(x)]$

$$\frac{\Delta y}{\Delta x} = f(x + \Delta x)\frac{\Delta g}{\Delta x} + g(x)\frac{\Delta f}{\Delta x} \quad \text{Product Rule} \quad \frac{dy}{dx} = f(x)\frac{dg}{dx} + g(x)\frac{df}{dx}$$

Example  $y = x^2 \sin x$  Product Rule  $\frac{dy}{dx} = x^2 \cos x + 2x \sin x$

A picture shows the two unshaded pieces of  $\Delta y = f(x + \Delta x)\Delta g + g(x)\Delta f$



Example  $f(x) = x^n$   $g(x) = x$   $y = f(x)g(x) = x^{n+1}$

$$\text{Product Rule} \quad \frac{dy}{dx} = x^n \frac{dx}{dx} + x \frac{dx^n}{dx} = x^n + x n x^{n-1} = (n+1)x^n$$

The correct derivative of  $x^n$  leads to the correct derivative of  $x^{n+1}$

**Quotient Rule** If  $y = \frac{f(x)}{g(x)}$  then  $\frac{dy}{dx} = \left( g(x)\frac{df}{dx} - f(x)\frac{dg}{dx} \right) / g^2$

$$\text{EXAMPLE} \quad \frac{d}{dx} \left( \frac{\sin x}{\cos x} \right) = (\cos x (\cos x) - \sin x (-\sin x)) / \cos^2 x$$

This says that  $\frac{d}{dx} \tan x = \frac{1}{\cos^2 x} = \sec^2 x$  (Notice  $(\cos x)^2 + (\sin x)^2 = 1$ )

$$\text{EXAMPLE} \quad \frac{d}{dx} \left( \frac{1}{x^4} \right) = \frac{x^4 \text{ times } 0 - 1 \text{ times } 4x^3}{x^8} = \frac{-4}{x^5} \quad \text{This is } nx^{n-1}$$

$$\text{Prove the Quotient Rule} \quad \Delta y = \frac{f(x + \Delta x)}{g(x + \Delta x)} - \frac{f(x)}{g(x)} = \frac{f + \Delta f}{g + \Delta g} - \frac{f}{g}$$

$$\text{Write this } \Delta y \text{ as } \frac{g(f + \Delta f) - f(g + \Delta g)}{g(g + \Delta g)} = \frac{g\Delta f - f\Delta g}{g(g + \Delta g)}$$

Now divide that  $\Delta y$  by  $\Delta x$  As  $\Delta x \rightarrow 0$  we have the Quotient Rule

**Practice Questions**

1. Product Rule: Find the derivative of  $y = (x^3)(x^4)$ . Simplify and explain.
2. Product Rule: Find the derivative of  $y = (x^2)(x^{-2})$ . Simplify and explain.
3. Quotient Rule: Find the derivative of  $y = \frac{\cos x}{\sin x}$ .
4. Quotient Rule: Show that  $y = \frac{\sin x}{x}$  has a maximum (zero slope) at  $x = 0$ .
5. Product and Quotient! Find the derivative of  $y = \frac{x \sin x}{\cos x}$ .
6.  $g(x)$  has a minimum when  $\frac{dg}{dx} = 0$  and  $\frac{d^2g}{dx^2} > 0$  The graph is bending up  
 $y = \frac{1}{g(x)}$  has a *maximum* at that point: Show that  $\frac{dy}{dx} = 0$  and  $\frac{d^2y}{dx^2} < 0$

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**Resource: Highlights of Calculus**  
Gilbert Strang

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