Product and Quotient Rules

Goal To find the derivative of $\boldsymbol{y = f(x)g(x)}$ from $\frac{df}{dx}$ and $\frac{dg}{dx}$

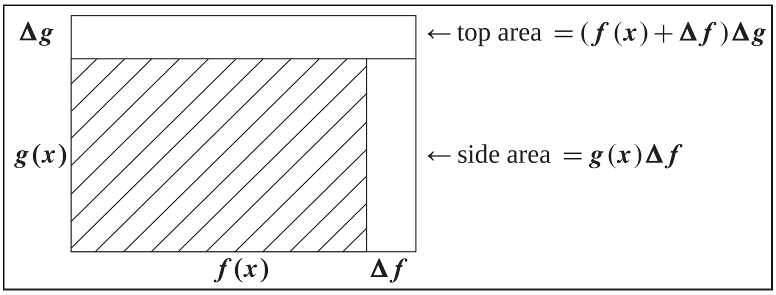
Idea Write $\Delta y = f(x+\Delta x)\,g(x+\Delta x) - f(x)g(x)$ by separating Δf and Δg

That same Δy is $f(x+\Delta x)\,[g(x+\Delta x)-g(x)]+g(x)\,[f(x+\Delta x)-f(x)]$

$\frac{\Delta y}{\Delta x} = f(x+\Delta x)\frac{\Delta g}{\Delta x} + g(x)\frac{\Delta f}{\Delta x}$ **Product Rule** $\boldsymbol{\frac{dy}{dx} = f(x)\frac{dg}{dx} + g(x)\frac{df}{dx}}$

Example $\quad y = x^2 \sin x \quad$ Product Rule $\frac{dy}{dx} = x^2 \cos x + 2x \sin x$

A picture shows the two unshaded pieces of $\Delta y = f(x+\Delta x)\Delta g + g(x)\Delta f$



Example $\quad f(x) = x^n \quad g(x) = x \quad \boldsymbol{y = f(x)g(x) = x^{n+1}}$

Product Rule $\frac{dy}{dx} = x^n \frac{dx}{dx} + x\frac{dx^n}{dx} = x^n + x\,n x^{n-1} = \boldsymbol{(n+1)x^n}$

The correct derivative of x^n leads to the correct derivative of x^{n+1}

Quotient Rule If $y = \frac{f(x)}{g(x)}$ then $\boldsymbol{\frac{dy}{dx} = \left(g(x)\frac{df}{dx} - f(x)\frac{dg}{dx}\right)\Big/ g^2}$

EXAMPLE $\frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = (\cos x(\cos x) - \sin x(-\sin x))\Big/\cos^2 x$

This says that $\boldsymbol{\frac{d}{dx}\tan x = \frac{1}{\cos^2 x} = \sec^2 x}$ (Notice $(\cos x)^2 + (\sin x)^2 = 1$)

EXAMPLE $\frac{d}{dx}\left(\frac{1}{x^4}\right) = \frac{x^4 \text{ times } 0 - 1 \text{ times } 4x^3}{x^8} = \boldsymbol{\frac{-4}{x^5}}$ This is nx^{n-1}

Prove the Quotient Rule $\quad \Delta y = \frac{f(x+\Delta x)}{g(x+\Delta x)} - \frac{f(x)}{g(x)} = \boldsymbol{\frac{f+\Delta f}{g+\Delta g} - \frac{f}{g}}$

Write this Δy as $\frac{g(f+\Delta f) - f(g+\Delta g)}{g(g+\Delta g)} = \boldsymbol{\frac{g\Delta f - f\Delta g}{g(g+\Delta g)}}$

Now divide that Δy by $\Delta x \quad$ As $\Delta x \to 0$ we have the Quotient Rule

Practice Questions

1. Product Rule: Find the derivative of $y = (x^3)(x^4)$. Simplify and explain.
2. Product Rule: Find the derivative of $y = (x^2)(x^{-2})$. Simplify and explain.
3. Quotient Rule: Find the derivative of $y = \dfrac{\cos x}{\sin x}$.
4. Quotient Rule: Show that $y = \dfrac{\sin x}{x}$ has a maximum (zero slope) at $x = 0$.
5. Product and Quotient! Find the derivative of $y = \dfrac{x \sin x}{\cos x}$.
6. $g(x)$ has a minimum when $\dfrac{dg}{dx} = 0$ and $\dfrac{d^2g}{dx^2} > 0$ The graph is bending up

   $y = \dfrac{1}{g(x)}$ has a *maximum* at that point: Show that $\dfrac{dy}{dx} = 0$ and $\dfrac{d^2y}{dx^2} < 0$