

Power Series and Euler's Formula

At $x = 0$, the n th derivative of x^n is the number $n!$ Other derivatives are 0.
Multiply the n th derivatives of $f(x)$ by $x^n/n!$ to match function with series

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$$f(x) = f(0) + f'(0)\frac{x}{1} + f''(0)\frac{x^2}{2} + \cdots + f^{(n)}(0)\frac{x^n}{n!} + \cdots$$

EXAMPLE 1 $f(x) = e^x$ All derivatives = 1 at $x = 0$ Match with $x^n/n!$

**Taylor Series
Exponential Series**

$$e^x = 1 + 1\frac{x}{1} + 1\frac{x^2}{2} + \cdots + 1\frac{x^n}{n!} + \cdots$$

EXAMPLE 2 $f = \sin x$ $f' = \cos x$ $f'' = -\sin x$ $f''' = -\cos x$

At $x = 0$ this is 0 1 0 -1 0 1 0 -1 REPEAT

$$\sin x = 1 \cdot \frac{x}{1} - 1\frac{x^3}{3!} + 1\frac{x^5}{5!} - \cdots \quad \text{ODD POWERS} \quad \sin(-x) = -\sin x$$

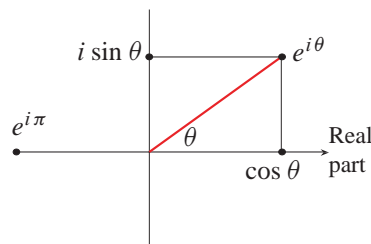
EXAMPLE 3 $f = \cos x$ produces 1 0 -1 0 1 0 -1 0 REPEAT

$$\cos x = 1 - 1\frac{x^2}{2!} + 1\frac{x^4}{4!} - \cdots \quad \text{EVEN POWERS} \quad \frac{d}{dx}(\cos x) = -\sin x$$

Imaginary $i^2 = -1$ and then $i^3 = -i$ **Find the exponential e^{ix}**

$$\begin{aligned} e^{ix} &= 1 + ix + \frac{1}{2!}(ix)^2 + \frac{1}{3!}(ix)^3 + \cdots \\ &= \left(1 - \frac{x^2}{2!} + \cdots\right) + i\left(x - \frac{x^3}{3!} + \cdots\right) \end{aligned} \quad \begin{array}{l} \text{Those are} \\ \cos x + i \sin x \end{array}$$

EULER'S GREAT FORMULA $e^{ix} = \cos x + i \sin x$



$$e^{i\theta} = \cos \theta + i \sin \theta$$

$$e^{i\theta} + e^{-i\theta} = 2 \cos \theta$$

$$e^{i\pi} = -1 \quad \text{combines 4 great numbers}$$

Two more examples of Power Series (Taylor Series for $f(x)$)

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \quad \text{"Geometric series"}$$

$$f(x) = -\ln(1-x) = \frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots \quad \text{"Integral of geometric series"}$$

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Resource: Highlights of Calculus
Gilbert Strang

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