Power Series and Euler's Formula

At $x = 0$, the nth derivative of x^n is the number $n!$ Other derivatives are 0.

Multiply the nth derivatives of $f(x)$ by $x^n/n!$ to match function with series

TAYLOR SERIES

$$f(x) = f(0) + f'(0)\frac{x}{1} + f''(0)\frac{x^2}{2} + \cdots + f^{(n)}(0)\frac{x^n}{n!} + \cdots$$

EXAMPLE 1 $f(x) = e^x$ All derivatives $= 1$ at $x = 0$ Match with $x^n/n!$

Taylor Series
Exponential Series

$$= \quad e^x = 1 + 1\frac{x}{1} + 1\frac{x^2}{2} + \cdots + 1\frac{x^n}{n!} + \cdots$$

EXAMPLE 2 $f = \sin x$ $f' = \cos x$ $f'' = -\sin x$ $f''' = -\cos x$

At $x = 0$ this is $0 \quad 1 \quad 0 \quad -1 \quad 0 \quad 1 \quad 0 \quad -1$ REPEAT

$$\mathbf{\sin x} = 1 \cdot \frac{x}{1} - 1\frac{x^3}{3!} + 1\frac{x^5}{5!} - \cdots$$

ODD POWERS $\sin(-x) = -\sin x$

EXAMPLE 3 $f = \cos x$ produces $1 \quad 0 \quad -1 \quad 0 \quad 1 \quad 0 \quad -1 \quad 0$ REPEAT

$$\mathbf{\cos x} = 1 - 1\frac{x^2}{2!} + 1\frac{x^4}{4!} - \cdots$$

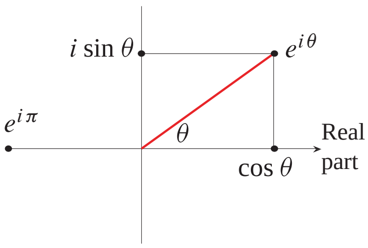
EVEN POWERS $\frac{d}{dx}(\cos x) = -\sin x$

Imaginary $i^2 = -1$ and then $i^3 = -i$ **Find the exponential e^{ix}**

$$e^{ix} = 1 + ix + \frac{1}{2!}(ix)^2 + \frac{1}{3!}(ix)^3 + \cdots$$
$$= \left(1 - \frac{x^2}{2!} + \cdots\right) + i\left(x - \frac{x^3}{3!} + \cdots\right)$$

Those are
$\cos x + i \sin x$

EULER'S GREAT FORMULA $e^{ix} = \cos x + i \sin x$



$e^{i\theta} = \cos\theta + i\sin\theta$

$e^{i\theta} + e^{-i\theta} = 2\cos\theta$

$e^{i\pi} = -1$ combines 4 great numbers

Two more examples of Power Series (Taylor Series for $f(x)$)

$$f(x) = \frac{\mathbf{1}}{\mathbf{1-x}} = 1 + x + x^2 + x^3 + \cdots$$ "Geometric series"

$$f(x) = -\mathbf{\ln(1-x)} = \frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$$ "Integral of geometric series"