Limits and Continuous Functions

Limits of $\Delta y/\Delta x$ are not the only limits in mathematics! Now we define limits more carefully. The goal is to achieve rigor without rigor mortis. These limits of a_n involve $n \to \infty$, not $\Delta x \to 0$:

1. $a_n = (n-3)/(n+3)$ (for large n, ignore the 3's and find $a_n \to 1$)
2. $a_n =$ probability of living to year n (unfortunately $a_n \to 0$)
3. $a_1 = .4$, $a_2 = .49$, $a_3 = .493$, ... *No matter what the remaining decimals are, the a's converge to a limit*. Possibly $a_n \to .493000\ldots$, but not likely.

The problem is to say what the limit symbol → really means.

A good starting point is to ask about convergence to *zero*. What does it mean to write $a_n \to 0$? The numbers $a_1, a_2, a_3, \ldots$, must become "small," but that is too vague. We will propose four definitions of ***convergence to zero***.

1. *All the numbers a_n are below* 10^{-10}. Not enough.
2. *The sequence is decreasing*. Not enough.
3. *For any small number you think of, at least one of the a_n's is smaller*. Not enough. $1, \frac{1}{2}, 1, \frac{1}{3}, 1, \frac{1}{4}, \ldots$, does not approach zero.
4. *For any small number you think of, the a_n's eventually go below that number and* ***stay below***. This is the correct definition.

To test for convergence to zero, start with a small number. The a_n's must go *below that number*. They may come back up and go below again—the first million terms make no difference. Eventually **all** terms must go below 10^{-10}. After waiting longer (possibly a lot longer), all terms drop below 10^{-20}.

Question 1 Does $10^{-3}, 10^{-2}, 10^{-6}, 10^{-5}, 10^{-9}, 10^{-8}, \ldots$ approach 0?
Answer Yes, These up and down numbers eventually stay below any ε.

Convergence to zero means that ***the sequence eventually goes below*** ε ***and stays there***. The smaller the ε, the tougher the test and the longer we wait. Think of ε as the tolerance, and keep reducing it.

To emphasize that ε comes from outside, Socrates can choose it. Whatever ε he proposes, the a's must eventually be smaller. *After some a_N, all the a's are below the tolerance ε*. Here is the exact statement:

$$\textbf{\textit{for any }} \varepsilon \textbf{\textit{ there is an }} N \textbf{\textit{ such that }} a_n < \varepsilon \textbf{\textit{ if }} n > N.$$

EXAMPLE $1, 0, \frac{1}{2}, 0, \frac{1}{3}, 0, \ldots$ approaches zero. These a's do not decrease steadily (the math word is "monotonically") but still their limit is zero. *Beyond a_{2001} all terms are below* $1/1000$. So $N = 2001$ for that ε.

Next we allow the numbers a_n to be *negative* as well as positive. They can converge upward toward zero, or they can come in from both sides.

The distance from zero is the absolute value $|a_n|$. *Therefore* $a_n \to 0$ *means* $|a_n| \to 0$.

for any ε ***there is an*** N ***such that*** $|a_n| < \varepsilon$ ***if*** $n > N$.

It is a short step to limits other than zero. ***The limit is*** L ***if the numbers*** $a_n - L$ ***converge to zero***. Our final test applies to the absolute value $|a_n - L|$:

$$\textbf{\textit{for any }} \varepsilon \textbf{\textit{ there is an }} N \textbf{\textit{ such that }} |a_n - L| < \varepsilon \textbf{\textit{ if }} n > N.$$

This is the definition of convergence! Only a finite number of a's are outside any strip around L. We write $a_n \to L$ or $\lim a_n = L$ or $\lim_{n\to\infty} a_n = L$.

The condition $a_{n+1} - a_n \to 0$ is ***not sufficient*** for convergence. However this condition is ***necessary***.

$$\textbf{\textit{If}}\ [a_n \text{ converges to } L]\ \textbf{\textit{then}}\ [a_{n+1} - a_n \text{ converges to zero}]. \tag{1}$$

Proof Because the a_n converge, there is a number N beyond which $|a_n - L| < \varepsilon$ and also $|a_{n+1} - L| < \varepsilon$. Since $a_{n+1} - a_n$ is the sum of $a_{n+1} - L$ and $L - a_n$, its absolute value cannot exceed $\varepsilon + \varepsilon = 2\varepsilon$. Therefore $a_{n+1} - a_n$ approaches zero.

Problem for you! Find a_n that do NOT CONVERGE even though $a_{n+1} - a_n$ does approach zero.

THE LIMIT OF $f(x)$ AS $x \to a$

The final step is to replace sequences by functions. The limit is taken as x approaches a specified point a (instead of $n \to \infty$). Example: As x approaches $a = 0$, the function $f(x) = 4 - x^2$ approaches $L = 4$. As x approaches $a = 2$, the function $5x$ approaches $L = 10$.

if x ***is close to*** a ***then*** $f(x)$ ***is close to*** L.

If $x - a$ is small, then $f(x) - L$ should be small. As before, the word *small* does not say everything. We really mean "arbitrarily small," or "below any ε." The difference $f(x) - L$ must become *as small as anyone wants*, when x gets near a. In that case $\lim_{x\to a} f(x) = L$. Or we write $f(x) \to L$ as $x \to a$.

The statement involves *two limits*. The limit $x \to a$ is forcing $f(x) \to L$. But it is wrong to expect the same ε in both limits. We cannot require that $|x - a| < \varepsilon$ produces $|f(x) - L| < \varepsilon$. ***It may be necessary to push*** x ***extremely close to*** a (closer than ε). We must guarantee that if x is close enough to a, then $|f(x) - L| < \varepsilon$.

We have come to the "***epsilon-delta definition***" of limits. First, Socrates chooses ε. He has to be shown that $f(x)$ is within ε of L, for every x near a. Then somebody else (maybe Plato) replies with a number δ. That gives the meaning of "near a." Plato's goal is to get $f(x)$ within ε of L, by keeping x within δ of a:

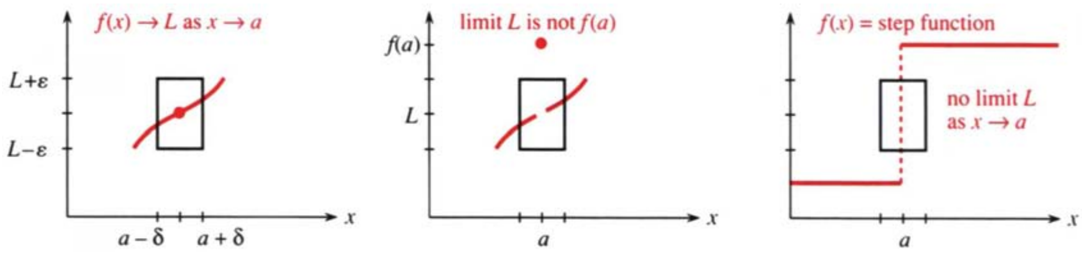
$$\textbf{\textit{if}} \quad 0 < |x - a| < \delta \quad \textbf{\textit{then}} \quad |f(x) - L| < \varepsilon. \tag{2}$$

The input tolerance is δ (delta), the output tolerance is ε. When Plato can find a δ for every ε, Socrates concedes that the limit is L.

EXAMPLE Prove that $\lim_{x \to 2} 5x = 10$. In this case $a = 2$ and $L = 10$.

Socrates asks for $|5x - 10| < \varepsilon$. Plato responds by requiring $|x - 2| < \delta$. What δ should he choose? In this case $|5x - 10|$ is exactly 5 times $|x - 2|$. So Plato picks δ below $\varepsilon/5$ (a smaller δ is always OK). Whenever $|x - 2| < \varepsilon/5$, multiplication by 5 shows that $|5x - 10| < \varepsilon$.

First, Socrates chooses the height of the box. It extends above and below L, by the small number ε. Second, Plato chooses the width. He must make the box narrow enough for the graph to go ***out the sides***. Then $|f(x) - L| < \varepsilon$.



S chooses height 2ε, then P chooses width 2δ. Graph must go out the sides.

When $f(x)$ has a jump, the box can't hold it. A step function has no limit as x approaches the jump, because the graph goes through the top or bottom—no matter how thin the box.

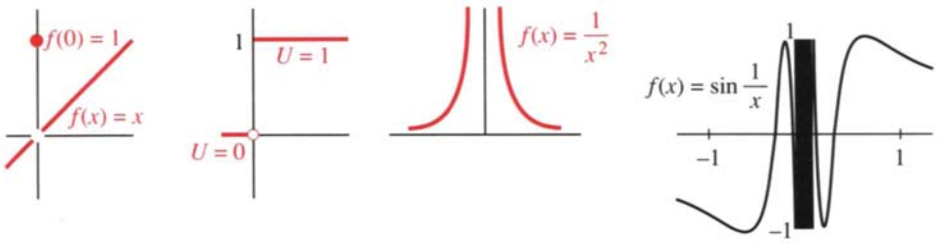
Continous Functions

The second figure has $f(x) \to L$, because in taking limits ***we ignore the final point*** $x = a$. The value $f(a)$ can be anything, with no effect on L. The first figure has more: $f(a)$ **equals** L. Then a special name applies—f is ***continuous***. The left figure shows a continuous function, the other figures do not.

May I summarize the usual (good) situation as x approaches a ?

1. The number $f(a)$ exists (f is defined at a)
2. The limit of $f(x)$ exists (it was called L)
3. The limit L equals $f(a)$ ($f(a)$ is the right value)

In such a case, $f(x)$ is ***continuous*** at $x = a$. These means $f(x) \to f(a)$ as $x \to a$. By way of contrast, these four functions are *not* continuous at $x = 0$.



The first function would be continuous if it had $f(0)=0$. But it has $f(0)=1$. After changing $f(0)$ to the right value, the problem is gone. The discontinuity is *removable*. Examples $2,3,4$ are more important and more serious. There is no "correct" value for $f(0)$:

2. $f(x)=$ step function (jump from 0 to 1 at $x=0$)
3. $f(x)=1/x^2$ (infinite limit as $x \to 0$)
4. $f(x)=\sin(1/x)$ (infinite oscillation as $x \to 0$).

EXAMPLE 1 $\sin x$ and $\cos x$ and all polynomials $P(x)$ are continuous functions.

EXAMPLE 2 The absolute value $|x|$ is continuous. Its slope jumps.

EXAMPLE 3 The function that jumps between 1 at fractions and 0 at non-fractions is ***discontinuous everywhere***. There is a fraction between every pair of non-fractions and vice versa. (Somehow there are many more non-fractions.)

DIFFERENTIABLE FUNCTIONS

The absolute value $|x|$ is continuous at $x=0$ but has no derivative. The same is true for $x^{1/3}$. ***Asking for a derivative is more than asking for continuity***. The reason is fundamental, and carries us back to the key definitions:

$$\textbf{\textit{Continuous}} \text{ at } x: \quad f(x+\Delta x)-f(x) \to 0 \text{ as } \Delta x \to 0$$

$$\textbf{\textit{Derivative}} \text{ at } x: \quad \frac{f(x+\Delta x)-f(x)}{\Delta x} \to f'(x) \text{ as } \Delta x \to 0.$$

In the first case, Δf goes to zero (maybe slowly). In the second case, Δf goes to zero *as fast as* Δx (because $\Delta f/\Delta x$ has a limit). That requirement is stronger:

At a point where $f(x)$ has a derivative, the function must be continuous. But $f(x)$ can be continuous with no derivative.

Here are two essential facts about ***a continuous function on a closed interval*** $a \leqslant x \leqslant b$. At the endpoints a and b, we require $f(x)$ to approach $f(a)$ and $f(b)$.

Extreme Value Property There are points $x_{\max}$ and $x_{\min}$ where $f(x)$ reaches its maximum value M and minimum value m.

$$f(x_{\max}) = M \geqslant f(x) \geqslant f(x_{\min}) = m \text{ for all } x \text{ in } [a,b].$$

Intermediate Value Property If F is between the minimum m and the maximum M, there is a point c between $x_{\min}$ and $x_{\max}$ where $f(c) = F$.

Practice problems on Limits

1. What is a_4 and what is the limit L?

   (a) $-1, +\frac{1}{2}, -\frac{1}{3}, \ldots$

   (b) $\frac{1}{2}, \frac{2}{4}, \frac{3}{8}, \ldots \; a_n = n/2^n$

   (c) $a_n = \sqrt[n]{n}$

   (d) $1+1, (1+\frac{1}{2})^2, (1+\frac{1}{3})^3, \ldots$

2. Show by example that these statements are false:

   (a) If $a_n \to L$ and $b_n \to L$ then $a_n/b_n \to 1$

   (b) If $a_n < 0$ and $a_n \to L$ then $L < 0$

   (c) If infinitely many a_n's are inside every strip around zero then $a_n \to 0$.

Find the limits if they exist. An $\varepsilon - \delta$ test is not required.

3. $\lim\limits_{t \to 2} \dfrac{t+3}{t^2 - 2}$

4. $\lim\limits_{x \to 0} \dfrac{|x|}{x}$

5. $\lim\limits_{x \to 1} \dfrac{\sin x}{x}$

6. $\lim\limits_{x \to a} [f(x) - f(a)]$ (?)

7. $\lim\limits_{x \to 0} \dfrac{\sin x}{\sin x/2}$

8. Which does the definition of a limit require?

   (1) $|f(x) - L| < \varepsilon \;\Rightarrow\; 0 < |x-a| < \delta$.

   (2) $|f(x) - L| < \varepsilon \;\Leftarrow\; 0 < |x-a| < \delta$.

   (3) $|f(x) - L| < \varepsilon \;\Leftrightarrow\; 0 < |x-a| < \delta$.

9. Give a correct definition of "$f(x) \to 0$ as $x \to \infty$."

10. The limit of $f(x) = \sin x$ as $x \to \infty$ does not exist. Explain why not.

11. *Important rule* As $x \to \infty$ the ratio of polynomials $f(x)/g(x)$ has the same limit as the ratio of their *leading terms*. $f(x) = x^3 + 2x^2 + 8x$ has leading term x^3 and $g(x) = 5x^6 + x$ has leading term $5x^6$. Therefore $f(x)/g(x)$ behaves like $x^3/5x^6 \to 0$, $g(x)/f(x)$ behaves like $5x^6/x^3 \to \infty$.

    Find the limit as $x \to \infty$ if it exists:

$$\frac{3x^2+2x+1}{3+2x+x^2} \quad \frac{x^4}{x^3+x^2} \quad \frac{x^2+1000}{x^3-1000} \quad x\sin\frac{1}{x}.$$

12. Suppose $f(x) = 1$ when x is a fraction m/n and $f(x) = 0$ when x is not a fraction (like e or π or $\sqrt{2}$). Why is this function not continuous at any point x ?

13. Explain in 110 words the difference between "we will get there if you hurry" and "we will get there only if you hurry" and "we will get there if and only if you hurry."

14. The statement "$3x \to 7$ as $x \to 1$" is false. Choose an ε for which no δ can be found. The statement "$3x \to 3$ as $x \to 1$" is true. For $\varepsilon = \frac{1}{2}$ choose a suitable δ.

15. *True or false*, with an example to illustrate:

    (a) If $f(x)$ is continuous at all x, it has a maximum value M.

    (b) If $f(x) \leqslant 7$ for all x, then f reaches its maximum.

    (c) If $f(1) = 1$ and $f(2) = -2$, then somewhere $f(x) = 0$.

    (d) If $f(1) = 1$ and $f(2) = -2$ and f is continuous on $[1,2]$, then somewhere on that interval $f(x) = 0$.

16. Explain with words and a graph why $f(x) = x\sin(1/x)$ is continuous but has no derivative at $x = 0$. Set $f(0) = 0$.

17. Create an $f(x)$ that is continuous only at $x = 0$.

Resource: Highlights of Calculus
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