## Inverse Functions and Logarithms

A function assigns an **output** $\boldsymbol{y = f(x)}$ to each **input** $\boldsymbol{x}$

A one-to-one function has different outputs $y$ for different inputs $x$

For the **inverse function** the input is $y$ and the output is $\boldsymbol{x = f^{-1}(y)}$

Example If $y = f(x) = x^5$ then $x = f^{-1}(y) = y^{\frac{1}{5}}$

KEY If $y = ax + b$ then solve for $x = \dfrac{y-b}{a}$ = inverse function

Notice that $\boldsymbol{x = f^{-1}(f(x))}$ and $\boldsymbol{y = f(f^{-1}(y))}$

The **chain rule** will connect the derivatives of $f^{-1}$ and $f$

The great function of calculus is $\boldsymbol{y = e^x}$

Its inverse function is the **"natural logarithm"** $\boldsymbol{x = \ln y}$

Remember that $x$ is the exponent in $y = e^x$

The rule $e^x e^X = e^{x+X}$ tells us that $\boldsymbol{\ln(yY) = \ln y + \ln Y}$

Add logarithms because you add exponents: $\boldsymbol{\ln(e^2 e^3) = 5}$

$(e^x)^n = e^{nx}$ (multiply exponent) tells us that $\boldsymbol{\ln(y^n) = n \ln y}$

We can change from base $e$ to base 10: New function $\boldsymbol{y = 10^x}$

The inverse function is the logarithm to base 10 Call it log: $\boldsymbol{x = \log y}$

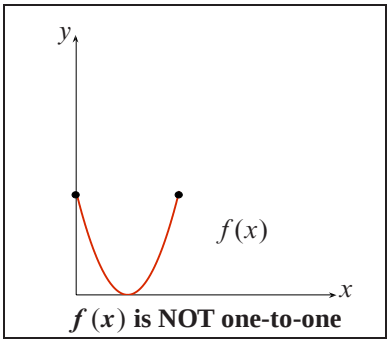
Then $\log 100 = 2$ and $\log \dfrac{1}{100} = -2$ and $\log 1 = 0$

We will soon find the beautiful derivative of $\ln y$ $\quad \boldsymbol{\dfrac{d}{dy}(\ln y) = \dfrac{1}{y}}$

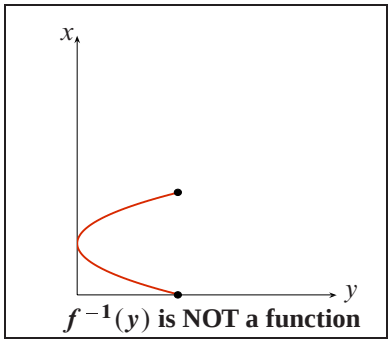
You can change letters to write that as $\boldsymbol{\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}}$

**Practice Questions**

1. What is $x = f^{-1}(y)$ if $y = 50x$ ?
2. What is $x = f^{-1}(y)$ if $y = x^4$ ? Why do we keep $x \geq 0$ ?
3. Draw a graph of an increasing function $y = f(x)$. This has different outputs $y$ for different $x$. **Flip the graph (switch the axes) to see $x = f^{-1}(y)$**
4. This graph has the same $y$ **from two $x$'s. There is no $f^{-1}(y)$**




**$f(x)$ is NOT one-to-one**




**$f^{-1}(y)$ is NOT a function**

5. The natural logarithm of $y = 1/e$ is $\ln(e^{-1}) = ?$ What is $\ln(\sqrt{e})$ ?
6. The natural logarithm of $y = 1$ is $\ln 1 = ?$ and also base 10 has $\log 1 = ?$
7. The natural logarithm of $(e^2)^{50}$ is ? The base 10 logarithm of $(10^2)^{50}$ is ?
8. I believe that $\ln y = (\ln 10)(\log y)$ because we can write $y$ in two ways $y = e^{\ln y}$ and also $y = 10^{\log y} = e^{(\ln 10)(\log y)}$. Explain those last steps.
9. Change from base $e$ and base 10 to **base 2**. Now $y = 2^x$ means $x = \log_2 y$. What are $\log_2 32$ and $\log_2 2$ ? Why is $\log_2(e) > 1$ ?

# Resource: Highlights of Calculus

Gilbert Strang