Growth Rates and Log Graphs

In order of fast growth as x gets large

$\log x$	x, x^2, x^3	$2^x, e^x, 10^x$	$x!, x^x$
logarithmic	polynomial	exponential	factorial

Choose $x = 1000 = 10^3$ so that $\log x = 3$ OK to use $x! \approx \dfrac{x^x}{e^x}$

$\log 1000 = 3$ $10^3, 10^6, 10^9$ $10^{300}, 10^{434}, 10^{1000}$ $10^{2566}, 10^{3000}$

Why is $1000^{1000} = 10^{3000}$? Logarithms are best for big numbers

Logarithms are exponents! $\log 10^9 = 9$ $\log \log x$ is VERY slow

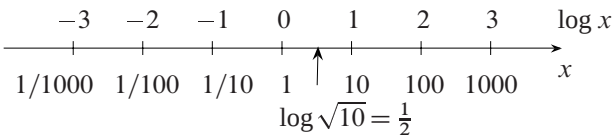
Logarithms **3, 6, 9** **300, 434, 1000** **2566, 3000**

Polynomial growth $\ll$ Exponential growth $\ll$ Factorial growth

Decay to zero for NEGATIVE powers and exponents

$\dfrac{1}{x^2} = x^{-2}$ decays much more slowly than the exponential $\dfrac{1}{e^x} = e^{-x}$

Logarithmic scale shows $x = 1, 10, 100$ equally spaced. NO ZERO!



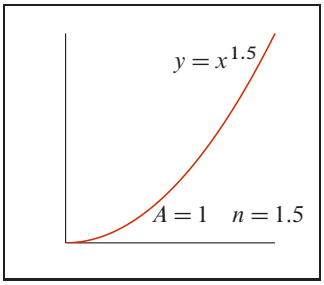
Question If $x = 1, 2, 4, 8$ are plotted, what would you see ?
Answer THEY ARE EQUALLY SPACED TOO!

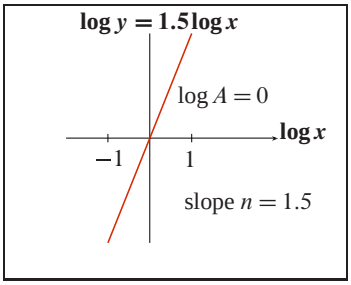
log-log graphs (log scale up and also across)

If $y = Ax^n$, how to see A and n on the graph ?

Plot $\log y$ versus $\log x$ to get a straight line

$\log y = \log A + n \log x$ **Slope on a log-log graph is the exponent n**





For $y = Ab^x$ use **semilog** (x versus $\log y$ is now a line) $\log y = \log A + x \log b$

Growth Rates and Log Graphs

New type of question How quickly does $\frac{\Delta f}{\Delta x}$ approach $\frac{df}{dx}$ as $\Delta x \to 0$?

The error $E = \frac{\Delta f}{\Delta x} - \frac{df}{dx}$ will be $E \approx A(\Delta x)^n$ What is n?

Usual one-sided $\frac{\Delta f}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$ only has $n = 1$

Centered difference $\frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x}$ has $n = 2$

Centered is much better than one-sided $E \approx (\Delta x)^2$ vs $E \approx \Delta x$

$\begin{bmatrix} \text{IDEA FOR} & f(x) = e^x \\ \text{PROJECT} & \text{at } x = 0 \end{bmatrix}$ One-sided E vs centered E
Graph log E vs log Δx Should see slope 1 or 2

Practice Questions

1. Does x^{100} grow faster or slower than e^x as x gets large?
2. Does $100 \ln x$ grow faster or slower than x as x gets large?
3. Put these in increasing order for large n:

   $\frac{1}{n}$, $n \log n$, $n^{1.1}$, $\frac{10^n}{n!}$

4. Put these in increasing order for large x:

   2^{-x}, e^{-x}, $\frac{1}{x^2}$, $\frac{1}{x^{10}}$

5. Describe the log-log graph of $y = 10x^5$ (graph log y vs log x)

Why don't we see $y = 0$ at $x = 0$ on this graph?

What is the slope of the straight line on the log-log graph?

The line crosses the vertical axis when $x =$ _____ and $y =$ _____

Then $\log x = 0$ and $\log y =$ _____

The line crosses the horizontal axis when $x =$ _____ and $y = 1$

Then $\log x =$ _____ and $\log y = 0$

6. Draw the semilog graph (a line) of $y = 10e^x$ (graph $\log y$ versus x)
7. That line cross the $x = 0$ axis at which $\log y$? What is the slope?

Resource: Highlights of Calculus

Gilbert Strang