

Differential Equations of Motion

A differential equation for $y(t)$ can involve dy/dt and also d^2y/dt^2

Here are examples with solutions C and D can be any numbers

$$\frac{d^2y}{dt^2} = -y \text{ and } \frac{d^2y}{dt^2} = -\omega^2 y \quad \text{Solutions } \begin{aligned} y &= C \cos t + D \sin t \\ y &= C \cos \omega t + D \sin \omega t \end{aligned}$$

Now include dy/dt and look for a solution method

$$m \frac{d^2y}{dt^2} + 2r \frac{dy}{dt} + ky = 0 \text{ has a damping term } 2r \frac{dy}{dt}. \quad \text{Try } y = e^{\lambda t}$$

Substituting $e^{\lambda t}$ gives $m\lambda^2 e^{\lambda t} + 2r\lambda e^{\lambda t} + ke^{\lambda t} = 0$

Cancel $e^{\lambda t}$ to leave the key equation for λ $m\lambda^2 + 2r\lambda + k = 0$

The quadratic formula gives $\lambda = \frac{-r \pm \sqrt{r^2 - km}}{m}$ Two solutions λ_1 and λ_2

The differential equation is solved by $y = Ce^{\lambda_1 t} + De^{\lambda_2 t}$

Special case $r^2 = km$ has $\lambda_1 = \lambda_2$ Then t enters $y = Ce^{\lambda_1 t} + Dte^{\lambda_1 t}$

EXAMPLE 1 $\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 8y = 0$ $m = 1$ and $2r = 6$ and $k = 8$

$$\lambda_1, \lambda_2 = \frac{-r \pm \sqrt{r^2 - km}}{m} \text{ is } -3 \pm \sqrt{9 - 8} \quad \text{Then } \begin{aligned} \lambda_1 &= -2 \\ \lambda_2 &= -4 \end{aligned}$$

Solution $y = Ce^{-2t} + De^{-4t}$ Overdamping with no oscillation

EXAMPLE 2 Change to $k = 10$ $\lambda = -3 \pm \sqrt{9 - 10}$ has $\begin{aligned} \lambda_1 &= -3 + i \\ \lambda_2 &= -3 - i \end{aligned}$

Oscillations from the imaginary part of λ **Decay** from the real part -3

Solution $y = Ce^{\lambda_1 t} + De^{\lambda_2 t} = Ce^{(-3+i)t} + De^{(-3-i)t}$

$e^{it} = \cos t + i \sin t$ leads to $y = (C + D)e^{-3t} \cos t + (C - D)e^{-3t} \sin t$

EXAMPLE 3 Change to $k = 9$ Now $\lambda = -3, -3$ (repeated root)

Solution $y = Ce^{-3t} + Dte^{-3t}$ includes the factor t

Practice Questions

1. For $\frac{d^2y}{dt^2} = 4y$ find two solutions $y = Ce^{at} + De^{bt}$. What are a and b ?
2. For $\frac{d^2y}{dt^2} = -4y$ find two solutions $y = C \cos \omega t + D \sin \omega t$. What is ω ?
3. For $\frac{d^2y}{dt^2} = 0y$ find two solutions $y = Ce^{0t}$ and (???)
4. Put $y = e^{\lambda t}$ into $2\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + y = 0$ to find λ_1 and λ_2 (**real** numbers)
5. Put $y = e^{\lambda t}$ into $2\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 3y = 0$ to find λ_1 and λ_2 (**complex** numbers)
6. Put $y = e^{\lambda t}$ into $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 0$ to find λ_1 and λ_2 (**equal** numbers)

Now $y = Ce^{\lambda_1 t} + Dte^{\lambda_1 t}$. The factor t appears when $\lambda_1 = \lambda_2$

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Resource: Highlights of Calculus
Gilbert Strang

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