Differential Equations of Motion

A differential equation for $y(t)$ can involve dy/dt and also d^2y/dt^2

Here are examples with solutions C and D can be any numbers

$$\frac{d^2y}{dt^2} = -y \text{ and } \boldsymbol{\frac{d^2y}{dt^2} = -\omega^2 y} \quad \text{Solutions} \quad \begin{array}{l} y = C \cos t + D \sin t \\ \boldsymbol{y = C \cos \omega t + D \sin \omega t} \end{array}$$

Now include dy/dt and look for a solution method

$$m\frac{d^2y}{dt^2} + 2r\frac{dy}{dt} + ky = 0 \text{ has a damping term } 2r\frac{dy}{dt}. \quad \textbf{Try } \boldsymbol{y = e^{\lambda t}}$$

Substituting $e^{\lambda t}$ gives $\quad m\lambda^2 e^{\lambda t} + 2r\lambda e^{\lambda t} + ke^{\lambda t} = 0$

Cancel $e^{\lambda t}$ to leave the key equation for λ $\quad \boldsymbol{m\lambda^2 + 2r\lambda + k = 0}$

The quadratic formula gives $\boldsymbol{\lambda} = \dfrac{-r \pm \sqrt{r^2 - km}}{m}$ Two solutions λ_1 and λ_2

The differential equation is solved by $\boldsymbol{y = Ce^{\lambda_1 t} + De^{\lambda_2 t}}$

Special case $r^2 = km$ has $\lambda_1 = \lambda_2$ Then t enters $y = Ce^{\lambda_1 t} + Dte^{\lambda_1 t}$

EXAMPLE 1 $\quad \dfrac{d^2y}{dt^2} + 6\dfrac{dy}{dt} + 8y = 0 \quad m = 1$ and $2r = 6$ and $k = 8$

$$\lambda_1, \lambda_2 = \frac{-r \pm \sqrt{r^2 - km}}{m} \text{ is } -3 \pm \sqrt{9 - 8} \quad \text{Then} \quad \begin{array}{l} \lambda_1 = \mathbf{-2} \\ \lambda_2 = \mathbf{-4} \end{array}$$

Solution $\quad \boldsymbol{y = Ce^{-2t} + De^{-4t}}$ $\quad$ Overdamping with no oscillation

EXAMPLE 2 $\quad$ Change to $k = 10$ $\quad \lambda = -3 \pm \sqrt{9 - 10}$ has $\begin{array}{l} \boldsymbol{\lambda_1 = -3 + i} \\ \boldsymbol{\lambda_2 = -3 - i} \end{array}$

Oscillations from the imaginary part of λ $\quad$ **Decay** from the real part -3

Solution $\quad y = Ce^{\lambda_1 t} + De^{\lambda_2 t} = Ce^{(-3+i)t} + De^{(-3-i)t}$

$e^{it} = \cos t + i \sin t$ leads to $y = (C + D)e^{-3t} \cos t + (C - D)e^{-3t} \sin t$

EXAMPLE 3 $\quad$ Change to $k = 9$ $\quad$ Now $\lambda = -3, -3$ (repeated root)

Solution $\quad y = Ce^{-3t} + \boldsymbol{Dte^{-3t}}$ includes the factor $\boldsymbol{t}$

Practice Questions

1. For $\dfrac{d^2y}{dt^2} = \mathbf{4}\boldsymbol{y}$ find two solutions $y = Ce^{at} + De^{bt}$. What are a and b ?
2. For $\dfrac{d^2y}{dt^2} = -\mathbf{4}\boldsymbol{y}$ find two solutions $y = C\cos\omega t + D\sin\omega t$. What is ω ?
3. For $\dfrac{d^2y}{dt^2} = \mathbf{0}\boldsymbol{y}$ find two solutions $y = Ce^{0t}$ and (???)
4. Put $y = e^{\lambda t}$ into $2\dfrac{d^2y}{dt^2} + 3\dfrac{dy}{dt} + y = 0$ to find λ_1 and λ_2 (**real** numbers)
5. Put $y = e^{\lambda t}$ into $2\dfrac{d^2y}{dt^2} + 5\dfrac{dy}{dt} + 3y = 0$ to find λ_1 and λ_2 (**complex** numbers)
6. Put $y = e^{\lambda t}$ into $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} + y = 0$ to find λ_1 and λ_2 (**equal** numbers)

   Now $y = Ce^{\lambda_1 t} + D\boldsymbol{t}e^{\lambda_1 t}$. The factor t appears when $\lambda_1 = \lambda_2$