DERIVATIVE OF ln y AND $\sin^{-1} y$

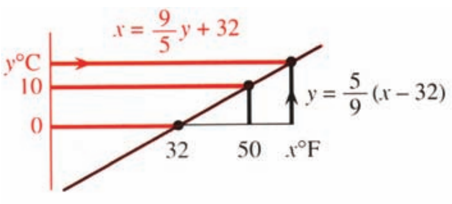
There is a remarkable special case of the chain rule. It occurs when $f(y)$ and $g(x)$ are "***inverse functions***." That idea is expressed by a very short and powerful equation: $f(g(x)) = x$. Here is what that means.

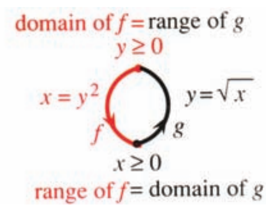
Inverse functions: Start with any input, say $x = 5$. *Compute* $y = g(x)$, say $y = 3$. Then compute $f(y)$, and *the answer must be* 5. What one function does, the inverse function undoes. If $g(5) = 3$ then $f(3) = 5$. ***The inverse function*** f ***takes the output*** y ***back to the input*** x.

EXAMPLE 1 $g(x) = x - 2$ and $f(y) = y + 2$ are inverse functions. Starting with $x = 5$, the function g subtracts 2. That produces $y = 3$. Then the function f adds 2. *That brings back* $x = 5$. To say it directly: ***The inverse of*** $y = x - 2$ ***is*** $x = y + 2$.

EXAMPLE 2 $y = g(x) = \frac{5}{9}(x - 32)$ and $x = f(y) = \frac{9}{5}y + 32$ are inverse functions (for temperature). Here x is degrees Fahrenheit and y is degrees Celsius. From $x = 32$ (freezing in Fahrenheit) you find $y = 0$ (freezing in Celsius). The inverse function takes $y = 0$ back to $x = 32$.

Notice that $\frac{5}{9}(x - 32)$ subtracts 32 *first*. The inverse $\frac{9}{5}y + 32$ adds 32 *last*. In the same way g multiplies last by $\frac{5}{9}$ while f multiplies first by $\frac{9}{5}$.





°F to °C to °F. Always $g^{-1}(g(x)) = x$ and $g(g^{-1} = (y)) = y$. If $f = g^{-1}$ then $g = f^{-1}$.

The inverse function is written $f = g^{-1}$ ***and pronounced*** "g inverse." ***It is not*** $1/g(x)$.

If the demand y is a function of the price x, then the price is a function of the demand. Those are inverse functions. ***Their derivatives obey a fundamental rule***: dy/dx ***times*** dx/dy ***equals*** 1. In Example 2, dy/dx is $5/9$ and dx/dy is $9/5$.

There is another important point. When f and g are applied in the *opposite order*, they still come back to the start. First f adds 2, then g subtracts 2. The chain $g(f(y)) = (y + 2) - 2$ brings back y. ***If*** f ***is the inverse of*** g ***then*** g ***is the inverse of*** f. The relation is completely symmetric, and so is the definition:

Inverse function: ***If*** $y = g(x)$ ***then*** $x = g^{-1}(y)$. ***If*** $x = g^{-1}(y)$ ***then*** $y = g(x)$.

The loop in the figure goes from x to y to x. The composition $g^{-1}(g(x))$ is the "identity function." Instead of a new point z it returns to the original x. This will make the chain rule particularly easy—leading to $(dy/dx)(dx/dy) = 1$.

EXAMPLE 3 $y = g(x) = \sqrt{x}$ and $x = f(y) = y^2$ are inverse functions.

Starting from $x = 9$ we find $y = 3$. The inverse gives $3^2 = 9$. The square of $\sqrt{x}$ is $f(g(x)) = x$. In the opposite direction, the square root of y^2 is $g(f(y)) = y$.

Caution That example does not allow x to be negative. The domain of g—the set of numbers with square roots—is restricted to $x \geqslant 0$. This matches the range of g^{-1}. The outputs y^2 are nonnegative. With *domain of* g = *range of* g^{-1}, the equation $x = (\sqrt{x})^2$ is possible and true. The nonnegative x goes into g and comes out of g^{-1}.

To summarize: ***The domain of a function matches the range of its inverse***. The inputs to g^{-1} are the outputs from g. The inputs to g are the outputs from g^{-1}.

If $g(x) = y$ ***then solving that equation for*** x ***gives*** $x = g^{-1}(y)$:

$$\text{if } y = 3x - 6 \quad \text{then } x = \tfrac{1}{3}(y+6) \quad (\text{this is } g^{-1}(y))$$
$$\text{if } y = x^3 + 1 \quad \text{then } x = \sqrt[3]{y-1} \quad (\text{this is } g^{-1}(y))$$

In practice that is how g^{-1} is computed: *Solve* $g(x) = y$. This is the reason inverses are important. Every time we solve an equation we are computing a value of g^{-1}.

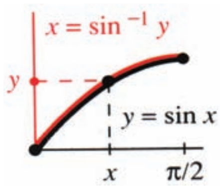
Not all equations have one solution. ***Not all functions have inverses***. For each y, the equation $g(x) = y$ is only allowed to produce one x. That solution is $x = g^{-1}(y)$. If there is a second solution, then g^{-1} will not be a function—because a function cannot produce two x's from the same y.

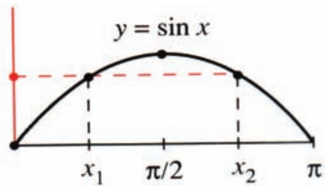
EXAMPLE 4 There is more than one solution to $\sin x = \frac{1}{2}$. Many angles have the same sine. On the interval $0 \leqslant x \leqslant \pi$, the inverse of $y = \sin x$ is not a function. The figure shows how two x's give the same y.

Prevent x from passing $\pi/2$ and the sine has an inverse. Write $x = \sin^{-1} y$.

The function g *has no inverse if two points* x_1 *and* x_2 *give* $g(x_1) = g(x_2)$. Its inverse would have to bring the same y back to x_1 and x_2. No function can do that; $g^{-1}(y)$ cannot equal both x_l and x_2. There must be only one x for each y.

To be invertible over an interval, g ***must steadily increase or steadily decrease***.





Inverse exists (one x for each y). No inverse function (two x's for one y).

THE DERIVATIVE OF g^{-1}

It is time for calculus. Forgive me for this very humble example.

EXAMPLE 5 (ordinary multiplication) The inverse of $y = g(x) = 3x$ is $x = f(y) = \frac{1}{3}y$. This shows with special clarity the rule for derivatives: ***The slopes*** $dy/dx = 3$ ***and*** $dx/dy = \frac{1}{3}$ ***multiply to give*** 1. This rule holds for all inverse functions, even if their slopes are not constant. It is a crucial application of the chain rule to the derivative of $f(g(x)) = x$.

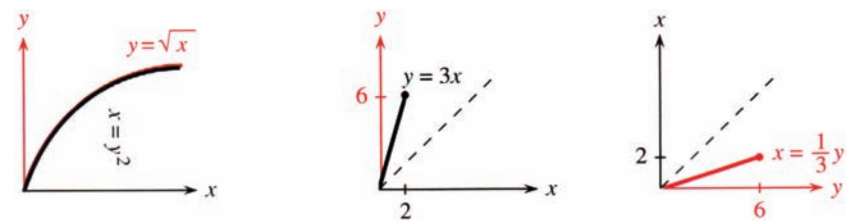
> (***Derivative of inverse function***) From $f(g(x)) = x$ the chain rule gives $f'(g(x))g'(x) = 1$. Writing $y = g(x)$ and $x = f(y)$, this rule looks better:
>
> $$\frac{dx}{dy}\frac{dy}{dx} = 1 \qquad \textit{or} \qquad \frac{dx}{dy} = \frac{1}{dy/dx}. \tag{1}$$
>
> The slope of $x = g^{-1}(y)$ times the slope of $y = g(x)$ equals one.

This is the chain rule with a special feature. Since $f(g(x)) = x$, *the derivative of both sides is* 1. If we know g' we now know f'. That rule will be tested on a familiar example. In the next section it leads to totally new derivatives.

EXAMPLE 6 The inverse of $y = x^3$ is $x = y^{1/3}$. We can find dx/dy two ways:

$$\text{directly}: \frac{dx}{dy} = \frac{1}{3}y^{-2/3} \qquad \text{indirectly}: \frac{dx}{dy} = \frac{1}{dy/dx} = \frac{1}{3x^2} = \frac{1}{3y^{2/3}}.$$

The equation $(dx/dy)(dy/dx) = 1$ is not ordinary algebra, but it is true. Those derivatives are limits of fractions. The fractions are $(\Delta x/\Delta y)(\Delta y/\Delta x) = 1$ and we let $\Delta x \to 0$.



Graphs of inverse functions: $x = \frac{1}{3}y$ is the mirror image of $y = 3x$.

Before going to new functions, I want to draw graphs. The figure shows $y = \sqrt{x}$ and $y = 3x$. What is special is that *the same graphs also show the inverse functions*. The inverse of $y = \sqrt{x}$ is $x = y^2$. The pair $x = 4, y = 2$ is the same for both. That is the whole point of inverse functions—if $2 = g(4)$ then $4 = g^{-1}(2)$. Notice that the graphs go steadily up.

The only problem is, the graph of $x = g^{-1}(y)$ is on its side. To change the slope from 3 to $\frac{1}{3}$, you would have to turn the figure. After that turn there is another problem—the axes don't point to the right and up. You also have to look in a mirror! (The typesetter refused to print the letters backward. He thinks it's crazy but it's not.) To keep the book in position, and the typesetter in position, we need a better idea.

The graph of $x = \frac{1}{3}y$ comes from ***turning the picture across the*** 45° ***line***. The y axis becomes horizontal and x goes upward. The point $(2,6)$ on the line $y = 3x$ goes into the point $(6,2)$ on the line $x = \frac{1}{3}y$. The eyes see a reflection across the 45° line. The mathematics sees the same pairs x and y. The special properties of g and g^{-1} allow us to know two functions—and draw two graphs—at the same time.[1] ***The graph of*** $x = g^{-1}(y)$ ***is the mirror image of the graph of*** $y = g(x)$.

EXPONENTIALS AND LOGARITHMS

The all-important example is $y = e^x$. Its inverse is the natural logarithm $x = \ln y$:

$$f^{-1}\big(f(x)\big) = \ln(e^x) = x \qquad f\big(f^{-1}(y)\big) = e^{\ln y} = y$$

We know that the derivative of e^x is e^x. So equation (1) will tell us the derivative of $x = \ln y$. This comes from the chain rule $(dx/dy)(dy/dx) = 1$.

$$\boxed{\frac{dx}{dy} = \frac{1}{dy/dx} = \frac{1}{e^x} = \frac{1}{y}}$$

The slope of $\ln y$ is therefore $1/y$. If you want to use different letters, there is nothing to stop you:

The function $f(x) = \ln x$ has slope $\frac{df}{dx} = \frac{1}{x}$.

We already knew the functions x^n/n with slope x^{n-1}, but $n = 0$ and slope x^{-1} was not allowed. Now we know that the natural logarithm fills this hole perfectly.

THE INVERSE OF A CHAIN $h(g(x))$

The functions $g(x) = x - 2$ and $h(y) = 3y$ were easy to invert. For g^{-1} we added 2, and for h^{-1} we divided by 3. Now the question is: If we create the composite function $z = h(g(x))$, or $z = 3(x - 2)$, what is its inverse?

Virtually all known functions are created in this way, from chains of simpler functions. *The problem is to invert a chain using the inverse of each piece*. The answer is one of the fundamental rules of mathematics:

[1] I have seen graphs with $y = g(x)$ and also $y = g^{-1}(x)$. For me that is wrong: it has to be $x = g^{-1}(y)$. If $y = \sin x$ then $x = \sin^{-1} y$.

The inverse of $z = h(g(x))$ is a chain of inverses ***in the opposite order***:

$$x = g^{-1}(h^{-1}(z)). \tag{2}$$

h^{-1} is applied first because h was applied last: $g^{-1}(h^{-1}(h(g(x)))) = x$.

That last equation looks like a mess, but it holds the key. In the middle you see h^{-1} and h. That part of the chain does nothing! The inverse functions cancel, to leave $g^{-1}(g(x))$. *But that is* x. The whole chain collapses, when g^{-1} and h^{-1} are in the correct order—which is opposite to the order of $h(g(x))$.

EXAMPLE 7 $z = h(g(x)) = 3(x-2)$ and $x = g^{-1}(h^{-1}(z)) = \frac{1}{3}z + 2$.

First h^{-1} divides by 3. Then g^{-1} adds 2. The inverse of $h \circ g$ is $g^{-1} \circ h^{-1}$. *It can be found directly by solving* $z = 3(x-2)$. A chain of inverses is like writing in prose—we do it without knowing it.

EXAMPLE 8 Invert $z = \sqrt{x-2}$ by writing $z^2 = x - 2$ and then $x = z^2 + 2$.

The inverse adds 2 and takes the square—*but not in that order*. That would give $(z+2)^2$, which is wrong. The correct order is $z^2 + 2$.

EXAMPLE 9 Inverse matrices $(AB)^{-1} = B^{-1}A^{-1}$ (this linear algebra is optional).

Suppose a vector x is multiplied by a square matrix B: $y = g(x) = Bx$. The inverse function multiplies by the ***inverse matrix***: $x = g^{-1}(y) = B^{-1}y$. It is like multiplication by $B = 3$ and $B^{-1} = 1/3$, except that x and y are vectors.

Now suppose a second function multiplies by another matrix A: $z = h(g(x)) = ABx$. The problem is to recover x from z. The first step is to invert A, because that came last: $Bx = A^{-1}z$. Then the second step multiplies by B^{-1} and brings back $x = B^{-1}A^{-1}z$. ***The product*** $B^{-1}A^{-1}$ ***inverts the product*** AB. The rule for matrix inverses is like the rule for function inverses—in fact it is a special case.

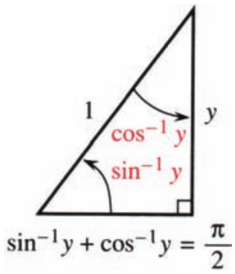
Mathematics is built on basic functions like the sine, and on basic ideas like the inverse. Therefore ***it is totally natural to invert the sine function***. The graph of $x = \sin^{-1} y$ is a mirror image of $y = \sin x$. This is a case where we pay close attention to the domains, since the sine goes up and down infinitely often. We only want *one piece* of that curve.

For the bold line the domain is restricted. *The angle* x *lies between* $-\pi/2$ *and* $+\pi/2$. On that interval the sine is increasing, so ***each*** y ***comes from exactly one angle*** x. If the whole sine curve is allowed, infinitely many angles would have $\sin x = 0$. The sine function could not have an inverse. By restricting to an interval where $\sin x$ is increasing, we make the function invertible.

The inverse function brings y back to x. It is $x = \sin^{-1} y$ (the ***inverse sine***):

$$x = \sin^{-1} y \text{ when } y = \sin x \text{ and } |x| \leqslant \pi/2. \tag{3}$$

The inverse starts with a number y between -1 and 1. It produces an angle $x = \sin^{-1} y$—***the angle whose sine is*** y. The angle x is between $-\pi/2$ and $\pi/2$, with the



Graphs of $\sin x$ and $\sin^{-1} y$. Their slopes are $\cos x$ and $1/\sqrt{1-y^2}$.

requiblack sine. Historically x was called the "arc sine" of y, and *arcsin* is used in computing. The mathematical notation is $\sin^{-1}$. *This has nothing to do with* $1/\sin x$.

The figure shows the 30° angle $x = \pi/6$. Its sine is $y = \frac{1}{2}$. ***The inverse sine of*** $\frac{1}{2}$ ***is*** $\pi/6$. Again: The symbol $\sin^{-1}(1)$ stands for the angle whose sine is 1 (this angle is $x = \pi/2$). We are seeing $g^{-1}(g(x)) = x$:

$$\sin^{-1}(\sin x) = x \text{ for } -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2} \qquad \sin(\sin^{-1} y) = y \text{ for } -1 \leqslant y \leqslant 1.$$

EXAMPLE 10 (important) If $\sin x = y$ find a formula for $\cos x$.

Solution We are given the sine, we want the cosine. The key to this problem must be $\cos^2 x = 1 - \sin^2 x$. When the sine is y, the cosine is the square root of $1 - y^2$:

$$\cos x = \cos(\sin^{-1} y) = \sqrt{1-y^2}. \tag{4}$$

This formula is crucial for computing derivatives. We use it immediately.

Problems

Read-through questions

Solve equations 1–6 for x, **to find the inverse function** $x = g^{-1}(y)$. **When more than one** x **gives the same** y, **write "no inverse."**

1. $y = 3x - 6$
2. $y = Ax + B$
3. $y = x^2 - 1$
4. $y = x/(x-1)$ [solve $xy - y = x$]
5. $y = 1 + x^{-1}$
6. $y = |x|$
7. Suppose f is increasing and $f(2) = 3$ and $f(3) = 5$. What can you say about $f^{-1}(4)$?

8. Suppose $f(2)=3$ and $f(3)=5$ and $f(5)=5$. What can you say about f^{-1}?

9. Suppose $f(2)=3$ and $f(3)=5$ and $f(5)=0$. How do you know that there is no function f^{-1}?

10. ***Vertical line test***: If no vertical line touches its graph twice then $f(x)$ is a ***function*** (one y for each x). ***Horizontal line test***: If no horizontal line touches its graph twice then $f(x)$ is ***invertible*** because _____ .

11. If $f(x)$ and $g(x)$ are increasing, which two of these might not be increasing?
    $f(x)+g(x)$ $f(x)g(x)$ $f(g(x))$ $f^{-1}(x)$ $1/f(x)$

12. If $y=1/x$ then $x=1/y$. If $y=1-x$ then $x=1-y$. The graphs are their own mirror images in the 45° line. Construct two more functions with this property $f=f^{-1}$ or $f(f(x))=x$.

13. If $dy/dx=1/y$ then $dx/dy=$ _____ and $x=$ _____ .

14. If $dx/dy=1/y$ then $dy/dx=$ _____ (these functions are $y=e^x$ and $x=\ln y$, soon to be honoblack properly).

15. The slopes of $f(x)=\frac{1}{3}x^3$ and $g(x)=-1/x$ are x^2 and $1/x^2$. Why isn't $f=g^{-1}$? What is g^{-1}? Show that $g'(g^{-1})'=1$.

Find dx/dy at the given point.

16. $y=\sin x$ at $x=\pi/6$

17. $y=\sin 2x$ at $x=\pi/4$

18. $y=\sin x^2$ at $x=3$

19. $y=x-\sin x$ at $x=0$

20. If y is a decreasing function of x, then x is a _____ function of y. Prove by graphs and by the chain rule.

21. If $f(x)>x$ for all x, show that $f^{-1}(y)<y$.

22. (a) Show by example that d^2x/dy^2 is not $1/(d^2y/dx^2)$.
    (b) If y is in meters and x is in seconds, then d^2y/dx^2 is in _____ and d^2x/dy^2 is in _____ .

23. Suppose the richest x percent of people in the world have $10\sqrt{x}$ percent of the wealth. Then y percent of the wealth is held by _____ percent of the people.

24. We know that $\sin\pi=0$. Why isn't $\pi=\sin^{-1}0$?

25. **True or false**, with reason:

(a) $(\sin^{-1} y)^2 + (\cos^{-1} y)^2 = 1$

(b) $\sin^{-1} y = \cos^{-1} y$ has no solution

(c) $\sin^{-1} y$ is an increasing function

(d) $\sin^{-1} y$ is an odd function

(e) $\sin^{-1} y$ and $-\cos^{-1} y$ have the same slope—so they are the same.

(f) $\sin(\cos x) = \cos(\sin x)$

Resource: Highlights of Calculus
Gilbert Strang