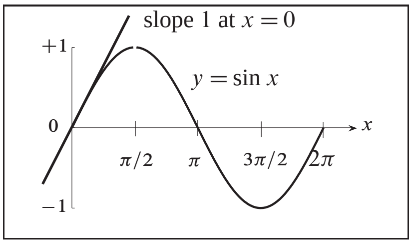
Derivative of the Sine and Cosine

This lecture shows that $\dfrac{d}{dx}(\sin x) = \cos x$ and $\dfrac{d}{dx}(\cos x) = -\sin x$

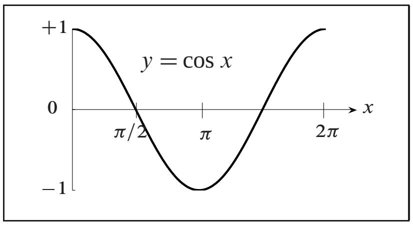
We have to measure the angle x in **radians** 2π radians = full 360 degrees

All the way around the circle (2π radians) **Length** $= \mathbf{2\pi}$ when the radius is 1
Part way around the circle (x radians) **Length** $= \boldsymbol{x}$ when the radius is 1



Slope $\cos x$

at $x = 0$	slope $1 = \cos 0$
at $x = \pi/2$	slope $0 = \cos \pi/2$
at $x = \pi$	slope $-1 = \cos \pi$



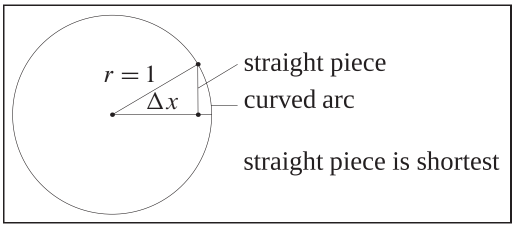
Slope $-\sin x$

at $x = 0$	slope $= 0 = -\sin 0$
at $x = \pi/2$	slope $-1 = -\sin \pi/2$
at $x = \pi$	slope $= 0 = -\sin \pi$

Problem: $\dfrac{\Delta y}{\Delta x} = \dfrac{\sin(x+\Delta x) - \sin x}{\Delta x}$ is not as simple as $\dfrac{(x+\Delta x)^2 - x^2}{\Delta x}$

Good idea to start at $x = 0$ Show $\dfrac{\Delta y}{\Delta x} = \dfrac{\sin \Delta x}{\Delta x}$ **approaches 1**

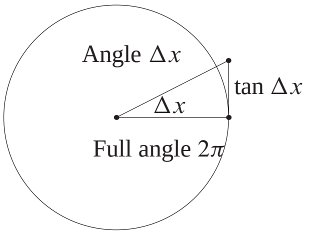
Draw a right triangle with angle Δx to see $\sin \Delta x \le \Delta x$



straight length $= \sin \Delta x$
curved length $= \Delta x$

IDEA $\dfrac{\sin \Delta x}{\Delta x} < 1$ and $\dfrac{\sin \Delta x}{\Delta x} > \cos \Delta x$ will **squeeze** $\dfrac{\sin \Delta x}{\Delta x} \to \mathbf{1}$ as $\Delta x \to 0$

To prove $\dfrac{\sin \Delta x}{\Delta x} > \cos \Delta x$ which is $\tan \Delta x > \Delta x$ **Go to a bigger triangle**



Triangle area $= \frac{1}{2}(1)(\tan \Delta x)$ greater than

Circular area $= \left(\dfrac{\Delta x}{2\pi}\right)$ (whole circle) $= \frac{1}{2}(\Delta x)$

The squeeze $\cos \Delta x < \dfrac{\sin \Delta x}{\Delta x} < 1$ tells us that $\dfrac{\sin \Delta x}{\Delta x}$ **approaches 1**

$\dfrac{(\sin \Delta x)^2}{(\Delta x)^2} < 1$ means $\dfrac{(1-\cos \Delta x)}{\Delta x}(1+\cos \Delta x) < \Delta x$

So $\dfrac{1-\cos \Delta x}{\Delta x} \to 0$ **Cosine curve has slope = 0**

For the slope at other x remember a formula from trigonometry: $\sin(x+\Delta x) = \sin x \cos \Delta x + \cos x \sin \Delta x$

We want $\Delta y = \sin(x+\Delta x) - \sin x$ Divide that by Δx

$$\frac{\Delta y}{\Delta x} = (\sin x)\left(\frac{\cos \Delta x - 1}{\Delta x}\right) + (\cos x)\left(\frac{\sin \Delta x}{\Delta x}\right) \qquad \text{Now let } \Delta x \to 0$$

In the limit $\dfrac{dy}{dx} = (\sin x)(0) + (\cos x)(1) = \cos x =$ Derivative of $\sin x$

For $y = \cos x$ the formula for $\cos(x+\Delta x)$ leads similarly to $\dfrac{dy}{dx} = -\sin x$

Practice Questions

1. What is the slope of $y = \sin x$ at $x = \pi$ and at $x = 2\pi$?
2. What is the slope of $y = \cos x$ at $x = \pi/2$ and $x = 3\pi/2$?
3. The slope of $(\sin x)^2$ is $2 \sin x \cos x$. The slope of $(\cos x)^2$ is $-2 \cos x \sin x$. Combined, the slope of $(\sin x)^2 + (\cos x)^2$ is **zero**. Why is this true ?
4. What is the **second derivative** of $y = \sin x$ (derivative of the derivative) ?
5. At what angle x does $y = \sin x + \cos x$ have zero slope ?

6. Here are amazing infinite series for $\sin x$ and $\cos x$. $\boldsymbol{e^{ix} = \cos x + i \sin x}$

$$\mathbf{sin}\,\boldsymbol{x} = \frac{x}{1} - \frac{x^3}{3\cdot 2\cdot 1} + \frac{x^5}{5\cdot 4\cdot 3\cdot 2\cdot 1} - \cdots \qquad (\textit{odd powers of } x)$$

$$\mathbf{cos}\,\boldsymbol{x} = 1 - \frac{x^2}{2\cdot 1} + \frac{x^4}{4\cdot 3\cdot 2\cdot 1} - \cdots \qquad (\textit{even powers of } x)$$

7. Take the derivative of the sine series to see the cosine series.

8. Take the derivative of the cosine series to see **minus** the sine series.

9. Those series tell us that for small angles $\boldsymbol{\sin x \approx x}$ and $\boldsymbol{\cos x \approx 1 - \frac{1}{2}x^2}$. With these approximations check that $(\sin x)^2 + (\cos x)^2$ is close to 1.

Resource: Highlights of Calculus
Gilbert Strang