Chain Rule for the Slope of $f(g(x))$

$y = g(x)$ $\quad z = f(y)$ $\quad \longrightarrow$ the chain is $\quad z = f(g(x))$

$y = x^5$ $\quad z = y^4$ $\quad \longrightarrow$ the chain is $\quad z = (x^5)^4 = x^{20}$

Average slope $\dfrac{\Delta z}{\Delta x} = \left(\dfrac{\Delta z}{\Delta y}\right)\left(\dfrac{\Delta y}{\Delta x}\right)$ Just cancel Δy

Instant slope $\dfrac{dz}{dx} = \dfrac{dz}{dy}\dfrac{dy}{dx}$ = **CHAIN RULE** (like cancelling dy)

You MUST change y to $g(x)$ in the final answer

Example of chain $\quad z = y^4 = (x^5)^4 \quad \dfrac{dz}{dy} = \mathbf{4y^3} \quad \dfrac{dy}{dx} = \mathbf{5x^4}$

Chain rule $\quad \dfrac{dz}{dx} = \left(\dfrac{dz}{dy}\right)\left(\dfrac{dy}{dx}\right) = (4y^3)(5x^4) = \mathbf{20y^3x^4}$

Replace y by x^5 to get only x $\quad \dfrac{dz}{dx} = 20(x^5)^3x^4 = \mathbf{20x^{19}}$

CHECK $\quad z = (x^5)^4 = x^{20}$ does have $\quad \dfrac{dz}{dx} = 20x^{19}$

1. Find $\dfrac{dz}{dx}$ for $z = \cos(4x)$ $\qquad$ Write $y = 4x$ and $z = \cos y$ so $\dfrac{dz}{dx} =$

2. Find $\dfrac{dz}{dx}$ for $z = (1+4x)^2$ $\qquad$ Write $y = 1+4x$ and $z = y^2$ so $\dfrac{dz}{dx} =$

CHECK $\quad (1+4x)^2 = 1+8x+16x^2$ so $\dfrac{dz}{dx} =$

Practice Questions

3. Find $\dfrac{dh}{dx}$ for $h(x) = (\sin 3x)(\cos 3x)$

Product rule first $\qquad$ Then the Chain rule for each factor

$$\frac{dh}{dx} = (\sin 3x)\frac{d}{dx}(\cos 3x) + (\cos 3x)\frac{d}{dx}(\sin 3x)$$

$$= (\sin 3x)(\text{CHAIN}) + (\cos 3x)(\text{CHAIN}) = \quad ?$$

Chain Rule for the Slope of $f(g(x))$

4. Tough challenge: Find the **second derivative** of $z(x) = f(g(x))$

FIRST DERIV	$\frac{dz}{dx} = \left(\frac{dz}{dy}\right)\left(\frac{dy}{dx}\right)$	Function of $y(x)$ times function of x
PRODUCT RULE	$\frac{d^2z}{dx^2} = \left(\frac{dz}{dy}\right)\frac{d}{dx}\left(\frac{dy}{dx}\right) + \left(\frac{dy}{dx}\right)\frac{d}{dx}\left(\frac{dz}{dy}\right)$	
SECOND DERIV	$\left(\frac{dz}{dy}\right)\left(\frac{d^2y}{dx^2}\right) + \left(\frac{dy}{dx}\right)\left(\frac{d^2z}{dy^2}\right)\left(\frac{dy}{dx}\right)$	$\frac{dy}{dx}$ *twice*!

Check $y = x^5$ $z = y^4 = x^{20}$ $\frac{dz}{dx} = 20x^{19}$ $\frac{d^2z}{dx^2} = 380x^{18}$

SECOND DERIV	$(4y^3)(20x^3) + (5x^4)(12y^2)(5x^4)$	$80 + 300 = 380$ OK

Resource: Highlights of Calculus

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