

## Chain Rule for the Slope of $f(g(x))$

1

### Chain Rule for the Slope of $f(g(x))$

$y = g(x)$     $z = f(y)$     $\rightarrow$  the chain is    $z = f(g(x))$   
 $y = x^5$     $z = y^4$     $\rightarrow$  the chain is    $z = (x^5)^4 = x^{20}$   
 Average slope    $\frac{\Delta z}{\Delta x} = \left(\frac{\Delta z}{\Delta y}\right) \left(\frac{\Delta y}{\Delta x}\right)$  Just cancel  $\Delta y$   
 Instant slope    $\frac{dz}{dx} = \frac{dz}{dy} \frac{dy}{dx} = \text{CHAIN RULE (like cancelling } dy)$   
 You MUST change  $y$  to  $g(x)$  in the final answer

Example of chain    $z = y^4 = (x^5)^4$     $\frac{dz}{dy} = 4y^3$     $\frac{dy}{dx} = 5x^4$   
 Chain rule    $\frac{dz}{dx} = \left(\frac{dz}{dy}\right) \left(\frac{dy}{dx}\right) = (4y^3)(5x^4) = 20y^3x^4$   
*Replace  $y$  by  $x^5$  to get only  $x$*     $\frac{dz}{dx} = 20(x^5)^3x^4 = 20x^{19}$   
 CHECK    $z = (x^5)^4 = x^{20}$  does have    $\frac{dz}{dx} = 20x^{19}$   
 1. Find  $\frac{dz}{dx}$  for  $z = \cos(4x)$    Write  $y = 4x$  and  $z = \cos y$  so  $\frac{dz}{dx} =$   
 2. Find  $\frac{dz}{dx}$  for  $z = (1 + 4x)^2$    Write  $y = 1 + 4x$  and  $z = y^2$  so  $\frac{dz}{dx} =$   
 CHECK    $(1 + 4x)^2 = 1 + 8x + 16x^2$  so  $\frac{dz}{dx} =$

### Practice Questions

3. Find  $\frac{dh}{dx}$  for  $h(x) = (\sin 3x)(\cos 3x)$

Product rule first   Then the Chain rule for each factor

$$\begin{aligned}
 \frac{dh}{dx} &= (\sin 3x) \frac{d}{dx}(\cos 3x) + (\cos 3x) \frac{d}{dx}(\sin 3x) \\
 &= (\sin 3x)(\text{CHAIN}) + (\cos 3x)(\text{CHAIN}) = ?
 \end{aligned}$$

Chain Rule for the Slope of  $f(g(x))$ 

4. Tough challenge: Find the **second derivative** of  $z(x) = f(g(x))$

FIRST DERIV	$\frac{dz}{dx} = \left(\frac{dz}{dy}\right) \left(\frac{dy}{dx}\right)$	Function of $y(x)$ times function of $x$
----------------	---	---

PRODUCT RULE	$\frac{d^2z}{dx^2} = \left(\frac{dz}{dy}\right) \frac{d}{dx} \left(\frac{dy}{dx}\right) + \left(\frac{dy}{dx}\right) \frac{d}{dx} \left(\frac{dz}{dy}\right)$
-----------------	---

SECOND DERIV	$\left(\frac{dz}{dy}\right) \left(\frac{d^2y}{dx^2}\right) + \left(\frac{dy}{dx}\right) \left(\frac{d^2z}{dy^2}\right) \left(\frac{dy}{dx}\right)$	$\frac{dy}{dx}$ twice!
-----------------	--	------------------------

**Check**  $y = x^5$     $z = y^4 = x^{20}$     $\frac{dz}{dx} = 20x^{19}$     $\frac{d^2z}{dx^2} = 380x^{18}$

SECOND DERIV	$(4y^3)(20x^3) + (5x^4)(12y^2)(5x^4)$	$80 + 300 = 380$ OK
-----------------	---------------------------------------	---------------------

MIT OpenCourseWare  
<http://ocw.mit.edu>

**Resource: Highlights of Calculus**  
Gilbert Strang

The following may not correspond to a particular course on MIT OpenCourseWare, but has been provided by the author as an individual learning resource.

For information about citing these materials or our Terms of Use, visit: <http://ocw.mit.edu/terms>.