Some Special Relativity Formulas

1 Introduction

The purpose of this handout is simple: to give you power in using special relativity! Even though you may not, at this stage, understand exactly where all of these formulas *come* from, you can certainly understand what they *mean* and have fun with them. Indeed, when you plug in some numbers, you can really get a feel for just how weird special relativity is.

2 Time Dilation

Suppose you're sitting on a bench, on a beautiful summer morning, watching the lovely trains pass by. Now suppose some train goes by at a speed of v, relative to you. Then, two events happen — lightning strikes and then a baby screams, say — and you measure the time interval between them to be t_0 on your watch. Suppose someone on the train also observes these two events, and she measures the time interval between them to be t on *her* watch. How are t and t_0 related? Contrary to Newtonian expectations, they are NOT EQUAL! In fact, as I showed in class, they are related by the following formula:

$$t = \frac{t_0}{\sqrt{1 - \frac{v^2}{c^2}}}, \tag{1}$$

where $c = 299{,}792{,}458$ meters/sec is the speed of light. (This is approximately 671,000,000 mph, for those of you who feel more comfortable with mph.) Since the quantity $\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ is always greater than 1 (you can check this for yourself), this means that, in your perspective, your watch ticks at a faster rate than the watch of somebody on the train! This effect — that *moving* *clocks run slow* — is known as **time dilation**. Notice that, as v approaches c, t approaches infinity! In other words, as a moving clock approaches the speed of light, the rate of its ticking becomes slower and slower (eventually infinitely slow) relative to you, the observer at rest.

Example: A super-train travels at 60 percent of the speed of light relative to you on your bench. Due to the extreme comfort of the bench, you accidentally doze off. Eventually, you wake up and determine that you napped for 4 hours! How long would a train observer measure your nap to be? Well, using the time dilation formula, here we have $t_0 = 4$ hrs and $v = 0.6c$, so $\frac{v}{c} = 0.6$. Therefore, a train observer measures your nap to last

$$t = \frac{4hrs}{\sqrt{1-(0.6)^2}} = 5hrs, \tag{2}$$

a whole hour longer than your watch said!

3 Length Contraction

As I mentioned in class, *length* is also relative. Suppose you measure the length of an object at rest to be l_0. Then, if that same object is moving at a speed of v relative to you, you'll measure its length to be l, where

$$l = l_0\sqrt{1 - \frac{v^2}{c^2}}. \tag{3}$$

Therefore, since $\sqrt{1 - \frac{v^2}{c^2}}$ is always *less* than 1, moving objects are *shorter* than they are at rest. In fact, the faster an object moves, the shorter it becomes, approaching zero length as its speed reaches the speed of light. Also, it's important to note that only *one* dimension of the object — the dimension in its direction of motion — gets contracted. The other two dimensions, which are perpendicular to the direction of motion, do not get contracted.

4 Relativistic Addition of Velocities

You return now to your train-watching festivities. Suppose a certain train moves at a velocity of v relative to you. Then, if an object — a baseball, for example — travels at a velocity of u relative to the *train*, the velocity that object will travel relative to *you* is given by

$$\frac{u+v}{1+\frac{uv}{c^2}}. \tag{4}$$

This is the so-called *relativistic addition of velocities* formula. Note that it is NOT $u+v$, as one would intuitively expect. (However, if u and v are much smaller than c, then you can show mathematically that this formula becomes approximately $u+v$, which is what we *do* expect for small speeds.)

Example: A (futuristic) rocket ship travels at a speed of 100,000,000 mph, moving in your left direction, relative to you. Relative to the rocket, a different rocket ship travels at a speed of 300,000,000 mph, in the direction *opposite* that which the original rocket ship is traveling. Question: How fast is the second rocket ship traveling relative to you? In this example, $v =$ 100,000,000 mph and $u = -300,000,000$ mph (negative because the second rocket is traveling in a direction opposite that of the first rocket). Plugging in numbers, we get

$$\frac{u+v}{1+\frac{uv}{c^2}} = \frac{(100,000,000-300,000,000)mph}{1+\frac{(100{,}000{,}000 mph)(-300{,}000{,}000 mph)}{(671{,}000{,}000 mph)^2}} \approx -214,000,000mph, \tag{5}$$

which means that the second rocket ship is traveling at a speed of 214,000,000 mph relative to you (and moving in your right direction). This is *approximately* what you'd get if you used the simple $u+v$ formula, but it differs by a noticeable amount. And the closer u and v are to c, the more that the relativistic addition of velocities formula will differ from the *non*-relativistic addition of velocities formula (which I called in class the *Galilean* addition of velocities).

5 Final Notes

Although all of the aforementioned effects are, in principle, always present in reality, it's only when speeds of objects reach a substantial fraction of the speed of light that the effects become *noticeable.* Also, remember that any inertial observer's perspective is just as good as any other. So, while you may say that the clocks on a moving train are running slow, people on the train will say that *your* clock is running slow, because *you're* the one in motion in their perspective. In other words, all of these effects are *reciprocal.* That said, it's extremely important always to specify *who's* doing the measurement of a certain quantity. It doesn't make sense, for example, to talk about "the" speed of an object. What *does* make sense is to talk about the speed of an object relative to one person, as well as the speed of the object relative to another person. (Of course, there is one exception to this particular example: all observers will measure the same speed for light, regardless of their relative motion. Thus, while one cannot sensibly speak of "the" speed of a baseball, one *can* sensibly speak of "the" speed of light.)