Combinatorics: The Fine Art of Counting

Week Five Problems

Before perusing this week's menu, please be sure that you have sampled a broad range of last week's problems and read through the solutions to check your understanding of the concepts. This week's problems build on the same theme – binomial coefficients in all their glory. Please begin with a few appetizers and then move on to the Binomial Buffet for a combinatorial feast.

Appetizers

1. A landscaper is planting a row of 15 trees including 5 birch trees and 10 pine trees. How many ways can this be done without having two birch trees next to each other?

    We will partition the 10 pine trees using the 5 birch trees as separators (note there are 6 distinct positions relative to the 5 birch trees). First place a pine tree between each pair of birch trees (4 pine trees total) to ensure no two birch trees are adjacent, then partition the remaining 6 using five separators which can be done in (6+5 5) = (11 5) = 462 different ways.

2. How many solutions does the equation $a+b+c+d+e+f = 2006$ have where a, b, c, d, e, and f are all positive integers?

    Given that the variables are all positive integers, we can subtract 1 from each of them and subtract 6 from the other side to obtain a new equation where we are expressing 2000 as an ordered sum of 6 non-negative integers. There are (2000+5 5) = (2005 5) ways to do this.

3. In how many ways can 20 identical balls be distributed among 5 distinct bins? In how many ways can 5 identical balls be distributed among 20 distinct bins?

    20 identical balls can be partitioned among 5 distinct bins using 4 separators in (20+4 4) = (24 4) = 10,626 ways. 5 identical balls can be partitioned among 20 distinct bins using 19 separators in (5+19 19) = (24 19) = (24 5) = 42,504 ways.

4. A piñata containing 10 Tootsie Rolls and 5 Hershey Kisses breaks open and the candy is eagerly collected by 8 children. In how many ways can the candy be distributed among the children?

    There are 8 children so we need 7 separators. There are (10+7 7) ways to distribute the Tootsie Rolls and (5+7 7) ways to distribute the Hershey Kisses, so the candy can be distributed in (17 7)(12 7) = 15,402,816 different ways.*

5. A drawer contains 8 distinct pairs of gloves each of which consists of a matching left and right glove. If you pull out four individual gloves at random, what is the probability that you have at least one pair of matching gloves? What is the probability you have two pairs of matching gloves?

   *There are (16 4) ways to select 4 of the 16 individual gloves. There are 16*14*12*10/4! ways to choose 4 non-matching gloves, so the probability of getting at least one pair of matching gloves is 1 – ((16*14*12*10)/4!) / (16 4) = 1 – 8/13 = **5/13**. There are (8 2) ways to choose two of the eight matching pairs, so the probability of getting two matching pairs is (8 2) / (16 4) = **1/65**.*

6. How many non-decreasing sequences of 7 decimal digits are there? (3334559 and 0223448 are two examples).

   *If we add a 0 to the beginning and a 9 to the end, every sequence of 7 non-decreasing decimal digits defines a unique sequence of 8 non-negative gaps which must sum to 9 (e.g. for 3334559 the gaps are 3+0+0+1+1+0+4+0 = 9, and for 0223448 the gaps are 0+2+0+1+1+0+4+1 = 9). We need 7 separators to count the partitions (the digits themselves separate the gaps), and we then find that there are **(9+7 7) = (16 7) = 11,440** of them.*

7. On an integer lattice, how many direct paths are there from the origin (0,0) to the point (n,n)? Now draw a diagonal line from (n,0) to (0,n). Every direct path from (0,0) to (n,n) must cross this line at some point of the form (k,n-k). How many direct paths from (0,0) to (n,n) go through the point (k,n-k)? Now write a combinatorial identity equating your answer to the first question to the sum of your answers to the second question.

   *There are **(2n n)** direct paths from (0,0) to (0,n). There are (n k) direct paths from (0,0) to (k,n-k), and (n n-k) direct paths from (k,n-k) to (n,n), so there are a total of **(n k)*(n n-k) = (n k)^2** direct paths from (0,0) to (0,n) that go through the point (k,n-k). Summing over all possible values of k we obtain the identity:*

   $$(n\ 0)^2 + (n\ 1)^2 + (n\ 2)^2 + \ldots + (n\ n)^2 = (2n\ n)$$

Binomial Buffet

The problems below all involve proving combinatorial identities for various summations that may arise in combinatorial problems. To make the problems easier/harder, you are only given the left hand side of the identity and have to determine the right hand side. In each case you should obtain a simple expression involving at most one or two binomial coefficients. For each problem, start by computing the sum for several small values and see if you can form a hypothesis as to what the right hand side might look like. Then try to prove a result either by using a combinatorial argument (e.g. a bijection), applying an identity you already know (e.g. the hockey stick), and/or using algebraic manipulations.

Keep an eye out for simple rearrangements that may make the problem easier, e.g. strategically changing (n k) to (n n-k), or using Gauss's addition trick – compute the sum forward and backwards. When constructing combinatorial arguments, bit-strings and groups of people (e.g. boys and girls) are useful things to count in different ways.

1. **(n 1) + 2*(n 2) + 3*(n 3) + …+ n*(n n) = n*2^{n-1}**

For symmetry reasons we will add the term 0*(n 0) = 0 to the LHS and then compute the sum twice, forwards and backwards:

0*(n 0) +   1*(n 1)   +   2*(n 2)   + … + n*(n n) +
n*(n n) + (n-1)*(n n-1) + (n-2)*(n n-2) + … + 0*(n 0)

Noting that (n k) = (n n-k), we can add the pairs in columns and obtain:

n*(n 0) + n*(n 1) + n*(n 2) + … + n(n n) = n*[(n 0) + (n 1) + (n 2) + … + (n n)] = n*2^{n}

We computed the sum on the LHS twice, so we now divide by 2 obtaining the RHS.

2. **(n 0)*(n 1) + (n 1)*(n 2) + (n 2)*(n 3) + … + (n n-1)*(n n) = (2n n-1)**

We will first re-write the LHS, strategically replacing (n k) with (n n-k) to obtain:

(n 0)*(n n-1) + (n 1)*(n n-2) + (n 2)*(n n-3) + … + (n n-1)*(n 0)

We can now see that each term represents a way to choose n-1 elements from a set of 2n elements. If we pick n-1 people from a group of n boys and n girls where we first pick the boys and then the girls (in all possible ways) we obtain the expression above which is equivalent to the LHS. But this is the same as simply picking n-1 people out of the entire group of 2n, which results in the RHS.

3. **(n 0) + (n+1 1) + (n+2 2) + … + (n+k k) = (n+k+1 k)**

If we re-write the LHS replacing (n k) with (n n-k) to obtain:

(n n) + (n+1 n) + (n+2 n) + … + (n+k n)

We can now apply the hockey stick identity to obtain (n+k+1 n+1) which is equivalent to the RHS.

4. **(n 0)^2 + (n 1)^2 + (n 2)^2 + … + (n n)^2 = (2n n)**

This identity was proven using block-walking on an integer lattice in problem #7 above. Alternatively, if we replace (n k)^2 with (n k)*(n n-k) on the LHS and consider choosing a group of n people out of a group of n boys and n girls by first choosing the boys and then choosing the girls (this is equal to the LHS) we can obtain the RHS by noting that this is the same as simply choosing n people out of a group of 2n people.

5. **(m 0)*(n k) + (m 1)*(n k-1) + (m 2)*(n k-2) + … + (m k)*(n 0) = (m+n k)**

Consider picking k people from a group of m boys and n girls. The LHS is equivalent to first picking the boys and then the girls (in all possible ways), while the RHS is equivalent to just picking k of the m+n people.

6. **(n k)*(k k) + (n k+1)*(k+1 k) + (n k+2)*(k+2 k) + … + (n n)*(n k) = (n k)*2^{n-k}**

Consider picking a team from n people with k starters and some number of substitutes (possibly 0). The LHS is equivalent to first picking a team of at least k people and then choosing k of them to be starters - each term represents a different team size ranging from k to n. The RHS is equivalent to picking exactly k starters plus an arbitrary subset of the remaining n-k people to fill out the team.

7. **(n 1) + 3*(n 3) + 5*(n 5) + … = n*2^{n-2}**   (sum over odd binomial coefficients)

Consider picking a team with an odd number of people with one of them designated as the captain out of a group of n people in total. The LHS is equivalent to picking the team and then picking one of them to be captain. The RHS represents first picking one of the n people to be the captain, and then choosing an even subset of the remaining n-1 people to fill out the team. It was shown in class that the number of even subsets is equal to the number of odd subsets, so half of all the 2^{n-1} subsets of n-1 people have an even number, or 2^{n-2}.

8. **(n 0)*(n k) + (n 1)*(n-1 k-1) + (n 2)*(n-2 k-2) + … + (n k)*(n-k 0) = (n k)*2^k**

Consider choosing a team of k people out of a group of n where each team member gets either a team hat or a team shirt. The LHS represents first choosing some number of people who get hats and then filling out the rest of the team with people who get shirts, whereas the RHS represents just choosing the entire team of k people and then letting each of the k team members choose a hat or a shirt..

9. *Multinomial Dessert Menu*

10. **What is the coefficient of the term $a^i b^j c^k$ in the expansion of $(a+b+c)^n$?**

*Before collecting like terms, each term in the expansion is a product of n factors each of which is **a**, **b**, or **c**. Those terms which have exactly i **a**'s, j **b**'s and k **c**'s can be combined into the single term $a^i b^j c^k$ and the number of such terms will be equal to the number of n letter strings containing i **a**'s, j **b**'s and k **c**'s, which by the Mississippi rule is **$n!/(i!*j!*k!)$**.*

11. **What is the sum of the coefficients of $(a+b+c)^n$?**

*Before collecting like terms each term is simply a product of n factors with three possibilities for each factor (a, b, or c), so there are 3^n terms and the sum of the coefficients is **3^n**. Collecting like terms doesn't change the sum of the coefficients.*

12. **Generalize your answers to problems 9 and 10 to more than three variables**.

*The multinomial coefficients for m variables are of the form $n!/(i_1!*i_2!*i_3!*\ldots*i_m!)$ and the sum of the coefficients is **m^n**.*