Combinatorics: The Fine Art of Counting

Week Four Solutions

1. An ice-cream store specializes in super-sized deserts. Their must famous is the "quad-cone" which has 4 scoops of ice-cream stacked one on top of the other. The order of scoops on a cone is distinct. They also offer a "quad-sundae" which has 4 scoops of ice-cream mixed together in a bowl. Once the scoops are in the bowl, you can't distinguish their order, you can only tell how many of each flavor there are. Note however that a sundae with 3 scoops of vanilla and 1 scoop of chocolate is different from a sundae with 3 scoops of chocolate and 1 scoop of vanilla. The store has 10 different flavors of ice cream to choose from.

    1. How many different quad-cones can you get?

       *10^4 = **10,000** - each cone is a sequence of 4 scoops of 10 possible flavors*

    2. How many have four distinct flavors?

       *10*9*8*7 = **5,040** - 10 choices for the first scoop, 9 for the second, etc...*

    3. How many different quad-sundaes can you get?

       *This is the most interesting case. The four scoops in a quad-sundae are indistinguishable (other than flavor), so we are being asked to categorize each of four identical objects into one of 10 distinct categories. This is a distinct partitioning problem. We need 9 separators for 10 flavors, so there are (9+4 9) = **(13 9) = (13 4) = 715** different quad-sundaes.*

    4. How many have four distinct flavors?

       ***(10 4) = 210** since we are just choosing a subset of four of the ten flavors.*

2. Suppose Gauss's school teacher had asked him to add up 1 + (1+2) + (1+2+3) + (1+2+3+4) + … + (1+2+3+…+100). What would the answer be?

   This is a summing problem – we are summing a bunch of sums (and if you consider each integer as a sum of 1's, we are summing sums of sums). Applying the hockey stick identity makes this easy to compute:
   *(2 2) + (3 2) + (4 2) + … + (101 2) = **(102 3) = 171,700***

   Now suppose Gauss really got into trouble and had to do the same thing for 100 more days in a row, only each day he had to go one sum further, i.e. the last number he added on the next day was 101, and so on until the 100^{th} day when he computed the sum of 1 + (1+2) + (1+2+3) + (1+2+3+4) + … + (1+2+3+…+200). The teacher than asked him to add up all the sums from all 100 days. What was his answer?

   Here we are summing 101 of our previous sums. We can't apply the hockey *stick immediately, because our sum doesn't start at (3 3) but we can consider the difference of two sums which do both start at (3 3):*

(102 3) + (103 3) + … + (202 3) = [(3 3) + … + (202 3)] – [(3 3) + … + (101 3)] = ***(203 4) – (102 4) = 64,435,475***

3. A donut shop sells donut holes in boxes with 24 donut holes per box. Assuming there are 8 different flavors of donut holes, how many distinct boxes of donut holes could you buy? (Assume you can't distinguish the arrangement of the donut holes inside the box).

   Consider 24 identical unflavored donut holes. We want to partition these 24 indistinguishable objects among 8 distinct flavors. We need 7 separators to do this, so we have (24+7 7) = ***(31 7) = 2,629,575*** *possible boxes.*

   Now suppose you look inside the box and notice you didn't get any donut holes in some of the flavors offered and insist that they give a new box with at least one donut hole of each flavor in it. How many distinct boxes of donut hole could they give you? What if you insist on at least two of each flavor?

   First put exactly one donut hole of each flavor in the box. The order doesn't matter, so there is only one distinct box with exactly one donut hole of each flavor. Then partition the remaining 16 donut holes among 8 distinct flavors. This gives (16+7 7) = ***(23 7) = 245,157*** *boxes with at least one of each flavor.*

   The same argument applies when we have at least two of each flavor, only there are only 8 donut holes left to partition after we put exactly two of each flavor in the box so there are (8+7 7) = ***(15 7) = 6,435*** *boxes with at least two of each flavor.*

   The donut shop also sells bagels by the half-dozen, with 8 different types to choose from. How many different bags of a half-dozen bagels could you buy?

   The fact that the number of objects being partitions (bagels) is less than the number of categories (flavors) doesn't change anything, so we still need 7 separators and get (6+7 7) = ***(13 7) = 1,716*** *possible bags.*

4. To celebrate the Holiday season, a local radio station decided to play the "Twelve days of Christmas" once on the first day of Christmas, twice on the second day, and so on playing the song 12 times on the 12^{th} day of Christmas. If we added up all the presents mentioned, how many presents would we there be in total?

   This is another summing problem, but we need to look carefully at the terms we are summing. The number of songs played each day changes, but the number of presents mentioned in each playing of the song is the same each time - we computed this in class as (14 3). To count the total number of presents mentioned we just need to add up the totals for each of the twelve days which gives us:

(14 3) + 2(14 3) + 3*(14 3) + … + 12*(14 3) = (1+2+3+…+12) *(14 3)*
*= **(13 2)*(14 3) = 28,392***

5. Plain Jane has 5 identical narrow rings that she likes to wear. She can wear them on any of her 8 fingers (but not her thumbs), and they are narrow enough that she can fit all 5 on one finger if she chooses to. How many different ways can Jane wear her rings? (note that Jane's rings may be plain, but she can tell her fingers apart).

   *(5+7 7) = **(12 7) = 792** ways if she wears all five. If she doesn't always wear all (or any) of them, she has (7 7) + (8 7) + … + (12 7) = **(13 8) = 1,287** choices.*

   If she puts at most one ring on each finger, how many ways are there for her to wear her rings?

   ***(8 5) = 56** if she wears all five since she is effective choosing a subset of 5 of her 8 fingers. There are (8 0) + (8 1) + (8 2) + … + (8 5) = **219** possibilities in all.*

   Suppose Jane is tired of being plain and paints her rings five different colors so she can tell them apart. How does this change your answers above?

   *Multiply the answers above by **5! = 120** - we can simply paint them after she has put them on, note that the order of painted rings is always distinct even if more than one are on the same finger.*

6. Sally lives in a city with a square grid of numbered streets which run east-west and numbered avenues that run north-south. Her house is located on the corner of 0^{th} Street and 0^{th} Avenue. Her aunt lives at the corner of 5^{th} St. and 3^{rd} Ave. How long is the shortest route (along streets or avenues) to her aunt's house? How many direct routes can Sally take to her aunt's house?

   *All direct routes to her aunt's house involve walking 5 blocks north and 3 blocks east. Thus the total number of direct routes is (5+3 3) = **(8 3) = 56***

   There is a grocery store at the corner of 2^{nd} St. and 2^{nd} Ave. If Sally needs to stop at the store on her way to her Aunt's, how many direct routes to her Aunt's house take her through the intersection of 2^{nd} St. and 2^{nd} Ave?

   *There are (2+2 2) direct routes Sally can take to the grocery store, after which there are (3+1 1) direct routes from the grocery store to her aunt's house, so there are a total of (2+2 2) * (3+1 1) = **(4 2) * (4 1) = 24***

   At her Aunt's house Sally hears on the radio that there has been an accident at the corner of 1^{st} St. and 2^{nd} Ave. Assuming she avoids this intersection, how many direct routes can she take home?

   *The simplest approach is to count the complement, i.e. count the number of direct routes through the intersection (as above), then subtract this from the total number of direct routes. There are (4+1 1) * (1+2 2) = (5 1)*(3 2) direct routes* *that go through the intersection of 1st St. and 2nd Ave and (8 3) direct routes in all, giving **(8 3) – (5 1) * (3 2) = 41** direct routes that avoid the intersection.*

7. Let S be the set of numbers {1,2,3,…,10}. Let A be a particular subset of S with 4 elements, e.g. {1,2,3,4}. Now let B be any subset of S with 4 elements chosen at random. What is the probability that B = A? What is the probability that A and B are disjoint? What is the probability that A and B overlap?

   *The probability that B=A is **1/(10 4) = 1/210**, the probability that A and B are disjoint is the same as the probability that B is a subset of A^c which is **(6 4)/(10 4) = 1/14**. The probability that A and B overlap is **1 – 1/14 = 13/14.***

   Now let B be a subset of S of any size chosen at random. What is the probability that B contains A? What is the probability that A contains B?

   *The number of subsets that contain A is equal to the number of subsets of A^c which is 2^6, so the probability that B contains A is **$2^6/2^{10}$ = 1/16**. The probability that A contains B is **$2^4/2^{10}$ = 1/64.***

8. Consider a tetrahedral stack of balls built by stacking a triangle with 1+2+…+10 balls on the bottom, 1+2+…+9 balls in the next layer, and so on with a single ball on top. How many balls are in this stack?. Now consider a square pyramid of balls constructed by stacking a 10x10 square of balls on the bottom, 9x9 balls in the next layer and so on with a single ball on top. How many balls are in this stack? Express your answer using binomial coefficients and then check it by computing $1^2 + 2^2 + \ldots + 10^2$.

   *For the tetrahedral stack we are simply summing triangle numbers. Applying the hockey stick identity gives us a total of (2 2) + (3 2) + … + (11 2) = **(12 3) = 165** balls in the tetrahedral stack.*

   *The square pyramid is composed of two tetrahedral stacks which share a common face (the tetrahedral stacks can be tilted so one face is vertical without changing the number of balls in the stack). There are **2*(12 3) – (11 2) = 385** balls in the square pyramid. $1^2 + 2^2 + \ldots + 10^2 = 385$ as expected*.

9. Consider the grid of points with integer coordinates in an x-y coordinate system. Now imagine an ant starting at the origin that can crawl up, down, left, or right at each step along a path, i.e. if the ant is at (a,b) it can move to (a-1,b), (a+1,b), (a,b-1), or (a,b+1) in the next step. Define an outward path to be a path that always takes the ant further away from the origin. Label each point with integer coordinates with the number of outward paths that lead from the origin to that point (it may be helpful to draw a picture). Compute the labels on the points (1,2), (-4,0), and (3,-2). Give a general formula for the label on the point (a,b) where a and b are arbitrary integers (possibly negative).

   We are clearly block walking here, but we now have 4 quadrants to consider instead of one. Note however that an outward path always stays in the same *quadrant, and the quadrants are symmetric. The point (a,b) has label **(|a|+|b| |a|)** or equivalently, **(|a|+|b| |b|)***

How many different points are possible end-points of an n step outward path?

For n > 0, there are 4n possible end-points on an n step outward path. For n = 0 there is just 1, the origin.

How many possible outward paths of length n are there in total?

For n > 0, the sum of (|a|+|b| |a|) over all a and b that satisfy |a|+|b| = n is just 4 times the sum in each quadrant, however if we include the axes in our count for the quadrant, we will count each point on an axis twice: there are four of these (0,n), (n, 0), (0, -n) and (-n, 0). This gives a total of $4*2^n - 4$ or $\mathbf{2^{n+2} - 4}$ (where n > 0). For n = 0 the total is just 1.

10. A children's stacking game contains a base with a pole in the center, 7 identical discs in each of 7 different sizes (49 discs total) with a hole in the middle of each disc, and a cap which screws on the top of the pole. The base is shaped so that only the biggest disc (size 7) will fit properly. Similarly the cap has a groove that will only accommodate the smallest disc (size 1). The object of the game is to create a complete stack of 7 discs with a size 7 disc on the bottom and a size 1 disc at the top, subject to the constraint that you can never stack a larger disc on top of a smaller one (e.g. 7654321, 7775331, and 7222211 are all complete stacks).

    How many different complete stacks are there?

    We will give three solutions, two using partitions, and one by block-walking.

    *If we think about the five discs in between the top and bottom disc, we are putting 5 otherwise identical discs into one of 7 categories based on size. Once the number of each size disc is chosen, a complete stack is determined – e.g. there is only one distinguishable complete stack with 3 size 5's, 1 size 4, and a 2, namely 7555421. Thus there are (5+6 6) = **(11 6) = 462** complete stacks since we need 6 separators to separate the 5 objects (discs) into 7 categories (sizes).*

    *An alternative approach is to consider the sum of the 6 gaps in sizes between the 7 discs from bottom to top (e.g. the stack 7775331 has gaps 0+0+2+2+0+2 = 6). These 6 distinctly ordered non-negative numbers must sum to 6. The number of ways of writing 6 as a sum of 6 non-negative numbers is (6+5 5) = **(11 5) = 462.** Here we are using 5 separators (+ signs) to separate 6 objects (1s) into 6 categories (distinct summands).*

    *A more visual approach is to turn the stack on its and project it onto a coordinate plane with the center pole pointing in the positive direction along the x-axis and the bottom of the bottom disk along the y-axis. If the edge of the bottom disk runs from (0,7) to (1,7) and the edge of the top disk runs from(6,1) to (7,1), every complete stack is equivalent to a direct route between (1,7) and (6,1). There are (5+6 6) = **(11 6) = 462** such routes.*

11. There are 10 positive integers less than 100 whose decimal digits sum to 9. (9, 18, 27, 36, 45, 54, 63, 72, 81, and 90). How many positive integers less than 10,000 have digits that sum to 9? How many have digits that sum to 10? (be careful here – no single digit can have the value 10). How about 15?

    If we consider any positive integer less than 10,000 as a sequence of 4 digits (add leading 0's as required) and insert '+' signs in between each of the digits, we obtain an expression containing four non-negative integers that sums to 9. Conversely, any such expression can be converted to a positive integer less than 10,000 – note that each summand must be less than or equal to 9 so they are all valid digits, and they can't all be zero so the number will be positive. Counting distinct partitions of 9 into 4 parts, we get (9+3 3) = ***(12 3) = 220*** *positive integers less than 10,000 whose digits sum to 9.*

    When the digits sum to 10 the only complication is we don't want to count distinct partitions of 10 where one of the summands is greater than 9. There are only 4 such cases (10+0+0+0, 0+10+0+0, 0+0+10+0, and 0+0+0+10), so we can simply subtract them from our count of all the distinct partitions of 10 into four parts obtaining (10+3 3) – 4 = ***(13 3) – 4 = 282****.*

    When the digits sum to 15, there are a lot more partitions we don't want to count, specifically any which have a summand greater than 9. At most one summand can be greater than 9 (this simplifies things considerably). If we simply pick one of the four summands to be greater than 9, we can then consider a partition of 5 into four parts and add 10 to the summand we picked to obtain a partition of 15 into four parts which has a summand greater than 9. Thus there are 4(5+3 3)= 4*(8 3) distinct partitions of 15 into four parts which include a summand greater than 9. Subtracting this from the total number of distinct partitions of 15 into four parts, (15+3 3) = (18 3), we obtain* ***(18 3) – 4*(8 3) = 592****.*

12. Consider a convex regular decagon (10-sided polygon) with vertices labeled A, B, C, D, E, F, G, H, I, J. How many different triangles can be formed among these vertices that do not share any edges with the decagon? Two triangles are distinct if they have different sets of vertices, otherwise they are the same.

    This problem can be solved in a couple of different ways, but they all require careful thought. Whatever method you use, check that your method gives the right answer if you apply it to a hexagon – there are exactly two triangles which do not share any edges with a labeled hexagon ({A,C,E} and {B,D,F}).

    Once you are satisfied with your solution, try to answer the same question for convex quadrilaterals, i.e. how many different convex quadrilaterals can be formed using the vertices of a labeled decagon that don't share any edges with the decagon? (Note that a set of n vertices that lie on the perimeter of any convex shape determines a unique convex polygon with n sides – there is exactly one convex quadrilateral for each subset of four vertices).

    Depending on which method you used to solve the triangle question, this problem will be either easy or quite tricky. As above, make sure that your approach gives the right answer on a labeled octagon where there are exactly two quadrilaterals which satisfy the problem.

Perhaps the most straight-forward approach is to count the total number of triangles among 10 vertices, (10 3), and then subtract those which share an edge with the decagon. A triangle can share either one or two edges with the decagon, so there are two cases to consider. There are 10 possible edges and 6 choices of a non-adjacent vertex that can form a triangle which shares one edge with the decagon. A triangle which share two edges with the decagon most contain three adjacent vertices, and there are 10 possibilities. Thus the total number of triangles which don't share an edge with the decagon is **(10 3) – 10*6 – 10 = 50**. Note that for a hexagon this approach give (6 3) – 6*2 – 6 = 2 as expected.

An alternative approach is to look at the size of the three gaps between the vertices that make up a triangle. If the triangle doesn't share an edge with the decagon, these gaps must all be greater than zero (i.e. no adjacent vertices) and the sum of the gaps must be 10-3=7. If we count the distinct partitions of 7 into 3 parts where every summand is non-zero, this will correspond to the gaps between a triangle which doesn't share an edge with the decagon. We can count partitions with non-zero summands by simply setting each summand to 1 and then partitioning what's left (7-3=4) among them. This give us a total of (4+2 2) distinct partitions of 7 into three non-zero parts. To get a particular triangle, we can choose a starting vertex (10 choices), and then determine the gaps by applying the partition in a particular direction (say clockwise) around the decagon. For example, starting at vertex A, the partition 2+2+3=7 would correspond to the triangle ADG. Finally we need to divide by three since we will count each triangle exactly three time using this approach (once for each choice of the starting vertex). This gives a total of **10*(4+2 2)/3 = 50**. Applying the same approach to a hexagon would yield **6*(0+2 2)/3 = 2** since we would wind up partitioning 0 after ensuring each gap was non-zero.

The second approach is slightly more involved than the first, but it generalizes more easily to the case of quadrilaterals. Now we have four gaps which must sum to 10-4=6 and since each must be non-zero, we want to count the number of distinct partitions of 6-4 = 2 into four parts which is (2+3 3). This yields **10*(5 3)/4 = 25.** In general, the number of convex k-gons that can be drawn on the vertices of an n-gon without sharing an edge is n*(n-k-1 k-1)/k.

The first approach to counting triangles can be generalized to counting quadrilaterals, but it is substantially more complex because a quadrilateral can share one or more edges with a decagon in several different ways each of which must be considered.