

Combinatorics: The Fine Art of Counting

Week Three Solutions

Note: in these notes multiplication is assumed to take precedence over division, so $4!/2!2! = 4!/(2!*2!)$, and binomial coefficients are written horizontally: $\binom{4}{2}$ denotes “4 choose 2” or $4!/2!2!$

Appetizers

1. How many 5 letter words have at least one double letter, i.e. two consecutive letters that are the same?

$$26^5 - 26*25^4 = 1,725,126$$

2. A Venn diagram is drawn using three circular regions of radius 1 with their centers all distance 1 from each other. What is the area of the intersection of all three regions? What is the area of their union?

Let A , B , and C be the three circular regions. $A \cup B \cup C$ can be seen to be equal to the union of three 60° sectors of radius 1, call them X , Y , and Z . The intersection of any 2 or 3 of these is an equilateral triangle with side length 1. Applying PIE, we have $|X \cup Y \cup Z| = |X| + |Y| + |Z| - (|X \cap Y| + |Y \cap Z| + |Z \cap X|) + |X \cap Y \cap Z|$ which is equal to $3*3*\pi/6 - 3*\sqrt{3}/4 + \sqrt{3}/4 = \pi/2 - \sqrt{3}/2$.

The intersection of any two of A , B , or C has area $2\pi/3 - \sqrt{3}/2$, as shown in class. Applying PIE, $|A \cup B \cup C| = |A| + |B| + |C| - (|A \cap B| + |B \cap C| + |C \cap A|) + |A \cap B \cap C|$ equal to $3\pi - 3*(2\pi/3 - \sqrt{3}/2) + (\pi/2 - \sqrt{3}/2) = 3\pi/2 + \sqrt{3}$

3. A diagonal of polygon is any line segment between vertices which is not an edge of the polygon. How many diagonals does an n -sided polygon have?

$$\binom{n}{2} - n$$

4. A diagonal of a polyhedron is any line segment between the vertices of a polyhedron which is not an edge of the polyhedron. A tetrahedron has no diagonals, while an octahedron has 3 diagonals. How many diagonals do the cube, dodecahedron and icosahedron have?

The cube has $\binom{8}{2} - 12 = 16$, the dodecahedron has $\binom{20}{2} - 30 = 160$, and the icosahedron has $\binom{12}{2} - 30 = 36$.

5. Let P_n denote the n^{th} centered pentagonal number: the number of points in a pentagonally symmetric lattice with one point in the center and n points on each side. The first four pentagonal numbers are 1, 6, 16, and 31. Find a formula for P_n .

$$P_n = 5*\binom{n}{2} + 1 \text{ (note that } \binom{1}{2} = 0, \text{ so this is correct when } n = 1)$$

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Week Three – Seasonal Menu

Winter

1. How many different even integers ≥ 4000 and < 7000 have four different digits? (AIME 1993 #1)

Partitioning the problem on the three possible values for the first digit (4,5, or 6) gives the sum $1 \cdot 4 \cdot 8 \cdot 7 + 1 \cdot 5 \cdot 8 \cdot 7 + 1 \cdot 4 \cdot 8 \cdot 7 = 728$.

2. How many integers less than 500 can be written as the sum of 2 positive integer cubes?

There are 7 positive cubes less than 500. The sum of any distinct pair is less than 500 except for $6^3 + 7^3$. These sums are all distinct. You can either check this or make use of the following bit of folklore: the first Ramanujan number (integers which can be written as the sum of two positive cubes in two different ways) is 1729.

There are thus $(7 \cdot 2) - 1 = 20$ integers less than 500 that can be written as the sum of two distinct positive integer cubes. If we permit the two cubes to be the same, then there are an additional six integers (note $7^3 + 7^3 > 500$) that can be written as the sum of two positive integer cubes for a total of 26.

3. Two of the squares of a 7x7 checkerboard are painted yellow and the rest are painted green. Two color schemes are indistinguishable if the board can be rotated so that they look the same. How many distinct color schemes are there? (AIME 1996 #7)

There are a total of $(49 \cdot 2)$ ways to pick two of the 49 squares to paint yellow, but these are not all distinct if we consider rotations indistinguishable. There are two cases to consider, paintings which have the two yellow squares diametrically opposing, and those which do not. There $48/2! = 24$ paintings of the first type (pick any square other than the center and the diametrically opposing square ($48 \cdot 1$ choices), then divide by 2 since we could have picked them in either order). There are $(49 \cdot 2) - 24 = 1,152$ paintings of the second type.

Each painting of the first type has two rotations which are indistinguishable color schemes, and each painting of the second type has four. Thus the total number of distinct color schemes is $24/2 + 1,152/4 = 300$.

4. Robert has 4 indistinguishable gold coins and 4 indistinguishable silver coins. Each coin has a face on one side and tails on the other. How many distinguishable stacks of the eight coins have no coins stacked face-to-face? (AIME 2005 #5)

Ignoring color for the moment, either all the coins are face down, or one of the 8 coins is the lowest coin which is face up and all the coins above it are also be face up. Thus there are $8+1 = 9$ ways to stack 8 colorless coins with none face-to-face. For any such stacking, we can choose any 4 of the 8 coins to be gold (the remaining 4 will be silver), giving a total of $9 \cdot \binom{8}{4} = 630$ distinct stacks.

5. Prove that it is impossible to draw a general Venn diagram for 4 sets with circles, i.e. 4 circles representing sets cannot be drawn in the plane so that every possible intersection of the 4 sets has its own distinct region with non-zero area.

To represent every possible intersection, there must be $2^4 = 16$ distinct regions in the Venn diagram. Pick any group of the 4 sets. There must be a region in the Venn diagram which corresponds to the elements which are in every one of the chosen sets, and not contained in any of the other sets (note that for the empty group this region corresponds to the points in U which are not in any of the four sets).

If we place a vertex at each point where the circumferences of two circles intersect, we can consider the Venn diagram a planar graph with edges defined by arcs between adjacent vertices along a circumference and the faces which are bounded by these edges correspond to the distinct regions of the Venn diagram with non-zero area (including the exterior face outside all the sets). Since every circle must intersect every other circle, this planar graph is clearly connected.

Two distinct circles can intersect in at most two points along their circumferences. Given that all possible intersections must have distinct non-zero areas, we may assume that each pair intersects in exactly two points and that these points are all distinct, thus each vertex has degree 4.

There are $2 \cdot \binom{4}{2} = 12$ vertices in the graph, and since therefore $12 \cdot 4/2 = 24$ edges. By Euler's formula for connected planar graphs, $V+F-E=2$, which implies that $F = 2 + E - V = 14$ which is less than 16. Thus it is not possible to draw a Venn diagram with 4 circles that has a distinct region for every possible intersection of sets.

Note that it is possible to make a general Venn diagram for 4 (or even more) sets using ellipses or polygons. The key difference is that unlike circles, these figures can intersect in more than two points while still being distinct.

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Week Three – Seasonal Menu

Spring

1. How many diagonals of a regular decagon (a 10 sided polygon) are not parallel to any of the sides (AMC12)?

A decagon has $\binom{10}{2} - 10 = 35$ diagonals. Each pair of opposing sides has three diagonals which are parallel to them, so $5 \cdot 3 = 15$ of the diagonals are parallel to a side. Therefore $35 - 15 = 20$ diagonals are not parallel to any side.

2. Let $p(m,n)$ denote the n^{th} centered m -agonal number: the number of points in an m -sided polygonal lattice with a point in the center and n points per side. Take $p(m,1)$ to be 1 (i.e. start with a single point) and then add concentric polygons with k equally spaced points for $2 \leq k \leq n$. Note that $p(3,n)$ is not the same as the n^{th} triangular number, nor is $p(4,n)$ the same as the n^{th} square number - these numbers are based on lattices constructed starting from a corner rather than the center. Find a general formula for $p(m,n)$ that works for any m and any n .

An m -sided centered polygonal lattice can be partitioned into m triangles with side length $n-1$ plus the center point. Thus $p(m,n) = m \cdot \binom{n-1}{2} + 1$.

3. In a shooting match a marksman must break eight targets arranged in three hanging columns of 3, 3, and 2 targets respectively. Whenever a target is broken, it must be the lowest unbroken target in its column. In how many different orders can the eight targets be broken? (AIME 1990 #8)

Label the columns A, B, and C. There is a 1-1 correspondence between distinct permutations of the string AAABBBCC and different valid orders for breaking the targets – each letter in the string indicates the next column to be targeted. Applying the Mississippi rule, there are $8!/3!3!2! = 560$ possible orders.

4. An integer is called *snakelike* if its decimal representation has consecutive digits alternately increasing and decreasing (e.g. 192837465 is snakelike). How many 4 digit numbers with distinct digits are snakelike? (AIME 2004 #6)

We can partition the problem into two cases based on whether the number contains the digit 0 or not. There are $\binom{9}{4}$ ways to choose 4 distinct non-zero decimal digits, and $\binom{9}{3}$ ways to choose 4 distinct decimal digits one of which is zero.

Given four distinct digits, for any of the $\binom{4}{2} = 6$ pairs of digits we can choose from these four, at most one snakelike number can be formed which has these two digits in the first two positions, since the smaller of the chosen pair must

come first, then the larger, and then the smaller of the remaining pair, followed by the last digit.

In the case where none of the digits is 0, a snakelike number can be formed for all but one of the 6 choices for the starting pair of digits – we can't choose the smallest two digits, since in this case the third digit will be greater than the second. Thus there are $(9\ 4) \cdot (6-1) = 630$ four digit snakelike numbers with distinct digits that don't contain 0.

In the case where one of the digits is 0, we can't choose this as one of the starting pair of digits, and we are left with $(3\ 2) = 3$ possible choices, all of which will result in a valid snakelike number, so there are $(9\ 3) \cdot 3 = 252$ four digit snakelike numbers with distinct digits that do contain 0.

Thus there are a total of $630 + 252 = 882$ four digit snakelike numbers with distinct digits.

5. A deck of forty cards consists of four 1's, four 2's, ..., and four 10's. One matching pair of cards is removed from the deck. Two cards are now drawn at random from the deck. What is the probability they form a pair? (AIME 2000B #3)

There are $(38\ 2)$ possible pairs of cards that can be drawn from the remaining deck, and $9 \cdot (4\ 2) + 1$ of these pairs are matching, thus the probability is given by the ratio $[9 \cdot (4\ 2) + 1] / (38\ 2) = 55/703$.

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Week Three – Seasonal Menu

Summer

1. A convex polyhedron P has 26 vertices, 60 edges, and 36 faces, 24 of which are triangular and 12 of which are quadrilaterals. A space diagonal is a line segment connecting two non-adjacent vertices that do not belong to the same face. How many space diagonals does P have? (AIME 2003 #3)

There are $\binom{26}{2}$ possible line segments between vertices of P . 60 of these are edges between adjacent vertices, and $2 \cdot 12 = 24$ are diagonals on one of the quadrilateral faces (note the triangular faces have 0 diagonals). Thus there are a total of $\binom{26}{2} - 60 - 24 = \mathbf{241}$ space diagonals.

2. The nine horizontal and nine vertical lines on an 8x8 checkerboard form r rectangles of which s are squares. Find r and s . (AIME 1997 #2)

The number of rectangles is $\binom{9}{2} \cdot \binom{9}{2} = \mathbf{1296}$ since each rectangle is determined by a pair of edges in distinct columns and a pair of edges in distinct rows. For a given side length j between 1 and 8, there is one square which has its lower left corner at position 0,0, and there are a total of $(9-j)^2$ locations which the lower left corner could occupy. Thus there is 1^2 8x8 square, 2^2 7x7 squares, ... 8^2 1x1 squares. This gives a total of $1^2 + 2^2 + 3^2 + \dots + 8^2 = \mathbf{204}$ squares (we will learn in a later class that the sum of the first n squares is $\frac{1}{6}n(n+1)(2n+1)$ or $\frac{1}{6}(10 \cdot 3) - \binom{9}{2}$ in this case).

3. The increasing sequence 2, 3, 5, 6, 7, 10, 11, ... consists of all positive integers which are neither the square nor the cube of a positive integer. Find the 500th term of this sequence. (AIME 1990 #1)

Since there are 22 squares less than 500, we know that the answer is at least 522. There are 8 positive cubes less than 521, so the answer could be as much as 531 if all the squares and cubes were distinct (note that we also have to skip $23^2 = 529$ if we get that far). We suspect the answer is probably 528 or 530, but we need to count carefully to avoid OBOEs.

*Let U be the set of positive integers less than 529. Let S be the subset of U which are squares and C the subset of U which are cubes. The integers in the defined sequence which are members of U is $(S \cup C)^c$. By the principle of inclusion/exclusion $|S \cup C| = |S| + |C| - |S \cap C|$. $S \cap C$ is the set of integers less than 529 which are both squares and cubes, which means they must be 6th powers. There are only two such numbers since $3^6 = 729$. Thus $|S \cup C| = 22 + 8 - 2 = 28$. So we have $|(S \cup C)^c| = |U| - |S \cup C| = 528 - 28 = 500$. The 500th term of the sequence is simply the largest element in $(S \cup C)^c$ which is **528**.*

4. Show why three-of-a-kind beats two-pair in poker by counting the number of poker hands containing exactly three cards of the same rank (with the other two cards different ranks) versus the number of poker hands containing two different pairs of cards with the same rank (but no three of the same rank).

*We can count these hands by first choosing ranks and then suit for each of the five cards. There are $\binom{13}{1}$ choices of rank for three cards of the same rank, and each of these three must be a different suit, so there are $\binom{4}{3}$ ways to choose the suits of these three cards. The remaining two card must have two different ranks chosen from the remaining 12 ranks, and each card may be one of four suits, giving a total of $\binom{13}{1} * \binom{4}{3} * \binom{12}{2} * \binom{4}{1} * \binom{4}{1} = 54,912$ hands with three-of-a-kind. (Note that we use binomial coefficients explicitly here to avoid careless mistakes – e.g. when choosing the ranks of the two remaining cards, the order of the choice does not matter).*

*Similarly for two-pair, there are $\binom{13}{2}$ ways to choose the two ranks of the two pairs, and the two cards in each pair must have two different suits chosen from the four suits. The fifth card may have any of the 11 remaining ranks and may be any of the four suits, giving $\binom{13}{2} * \binom{4}{2} * \binom{4}{2} * \binom{11}{1} * \binom{4}{1} = 123,552$.*

5. Ten points in the plane are given, no three co-linear. Four distinct segments joining pairs of these points are chosen at random with uniform probability. What is the probability that three of the four segments chosen form the sides of a triangle? (AIME 1999 #10)

There are $\binom{10}{2} = 45$ distinct line segments between pairs of points. There are thus $\binom{45}{4}$ ways to pick four distinct segments.

*There are $\binom{10}{3}$ distinct triangles that can be formed among the 10 points and each has three sides. A choice of four segments which contains a triangle must consist of the three sides of one of these $\binom{10}{3}$ triangles along with one of the other $45 - 3 = 42$ segments. Thus there are $\binom{10}{3} * 42$ ways of picking four distinct segments which include the sides of a triangle.*

*The probability is therefore $\binom{10}{3} * 4 / \binom{45}{4} = 16/473$.*

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Week Three – Seasonal Menu

Fall

1. Ten male/female couples meet for a dinner party and they all greet each other in the following manner: the men all shake hands, the women all exchange kisses, and each man exchanges kisses with each woman. Assuming every person greets every other person and counting each handshake or kiss exchange between two people just once, how many handshakes and kiss exchanges are there? How many greetings all together? Given that the number of greetings is the sum of the handshakes and the kisses, can you derive a general combinatorial identity?

There are $(10\ 2)$ handshakes, $(10\ 2) + 10^2$ kiss exchanges, and $(20\ 2)$ greetings in total. Since the first two numbers must sum to the third, we have the general combinatorial identity: $(2n\ 2) = 2(n\ 2) + n^2$. This can be easily verified algebraically, but it need not be – this is a perfectly valid combinatorial proof.*

2. Ten points are marked on a circle. How many distinct convex polygons can be drawn using some (or all) of the ten points as vertices? (Polygons are distinct unless they have exactly the same vertices.) (AIME 1989 #2)

Any subset of 3 or more vertices determines a unique convex polygon. There are $2^{10} - (10\ 2) - (10\ 1) - (10\ 0) = 1,024 - 45 - 10 - 1 = 968$ such subsets.

3. A fair coin is tossed ten times. Find the probability that heads never occurs on two consecutive tosses (AIME 1990 #9)

There are $2^{10} = 1024$ possible sequences of coin tosses, all equally likely. A sequence of ten coin tosses with no consecutive heads corresponds to a string of 10 bits with no adjacent 1s (and vice versa). It was shown in class that the number of n -bit strings with no adjacent 1s satisfies the Fibonacci recurrence: $F(n) = F(n-1) + F(n-2)$ starting with $F(1) = 2$ and $F(2) = 3$. Since $F(10) = 144$, the probability that heads never occurs on consecutive tosses is $144/1024 = 9/64$.

4. One hundred concentric circles with radii 1,2,3,...,100 are drawn in the plane. The inner circle is colored red and each region bounded by concentric circles is colored green or red with no two adjacent regions the same color. What fraction of the entire circle with radius 100 is colored green? (AIME 2003 #2)

We can compute the area colored by repeatedly including and excluding the areas of the concentric circles. This gives the sum $100^2\pi - 99^2\pi + 98^2\pi - 97^2\pi + \dots + 2^2\pi - 1^2\pi = \pi[(100^2-99^2) + (98^2-97^2) + \dots + (2^2-1^2)]$. Noting that the difference of two squares can be factored as $a^2-b^2 = (a-b)(a+b)$ and $(a-b) = 1$ in each pair of squares above, this simplifies to $\pi[(100+99)+(98+97)+\dots+(2+1)]$

which is simply $(101\ 2)*\pi$. The fraction of the entire circle which is colored green is thus $(101\ 2)*\pi / 100^2\pi = \mathbf{101/200}$.

5. Given n lines in the plane in general position (each line intersects every other line in a distinct point), into how many regions do they divide the plane?

We will give two separate proofs. The first uses Euler's formula for planar graphs. Place a vertex at each of the $(n\ 2)$ intersection points and then draw a circle in the plane which contains all these points in its interior and place a vertex at the $2n$ intersections of the lines with the circle. The number of regions contained inside the circle is equal to the number of regions of the plane partitioned by the lines. Now consider the connected planar graph formed by the $V = (n\ 2) + 2n$ vertices and the line segments or arcs between them. Each of the $(n\ 2)$ vertices at the intersection of two lines has degree 4, while the $2n$ vertices along the circle have degree 3. Counting edge-vertex combinations we have that $2E = 4(n\ 2) + 3*2n$, so $E = 2*(n\ 2) + 3n$. Applying Euler's formula $V+F-E=2$ to compute F we obtain $F = 2 + E - V = 2 + 2*(n\ 2) + 3n - (n\ 2) - 2n = 2 + (n\ 2) + n$. Since we only want to count faces in the interior of the circle, we subtract 1 for the exterior face to obtain a total of $(n\ 2) + n + 1$ regions.*

A simpler but more abstract proof is the following. Place a temporary test line which lies to one side of all the intersection points of the n lines and is not parallel to any pair of intersection points. This test line is intersected by all n lines since it is not parallel to any of them, and therefore it passes through $n+1$ regions of the plane partitioned by the n lines. Slide the test line across the plane, always keeping it parallel to its original position. As we do this the test line will encounter the $(n\ 2)$ intersection points of the n lines one by one, and each time it does it will enter exactly one new region of the plane (note that any intersection point involves only 2 lines and the test line already passes through 3 of the 4 regions determined by just these 2 lines). There are thus a total of $(n\ 2) + n + 1$ regions.

The second proof can be easily generalized to higher dimensions, e.g. intersecting planes in space.