Combinatorics: The Fine Art of Counting

Week Two Solutions

Note: In these notes 4!/2!2! means 4! divided by the quantity (2! * 2!), and binomial coefficients are written horizontally, i.e. (4 2) denotes 4 choose 2.

1. How many different ways can you rearrange the letters in BOSTON? How about MASSACHUSETTS?

   *Applying the Mississippi rule, BOSTON can be rearranged in 6!/2! = **360** distinct ways and MASSACHUSETTS can be rearranged in 13!/4!2!2! = **64,864,800** distinct ways.*

2. 20 students show up to HSSP looking for open classes. Only 3 classes are still open, one has 3 spots, one has 11 spots, and one has 6 spots. How many different ways can the students be arranged in the 3 classes?

   *(20 3)(17 11)(6 6) = 20!/3!11!6! = **14,108,640***

3. How many license plates with 3 decimal digits followed by 3 letters do not contain both the number 0 and the letter O?

   *Assuming all decimal digits and letters are permitted, there are a total of $10^3 * 26^3$ possible license plates. There are (10^3-9^3) strings of decimal digits that contain the number 0 and 26^3-25^3 strings of 3 letters that contain the letter O, so there are $(10^3-9^3)*(26^3-25^3)$ license plates that can formed from these strings. Thus there $10^3 * 26^3 - (10^3-9^3)*(26^3-25^3)$ = **17,047,279** license plates that do not contain both the number 0 and the letter O.*

4. How many ways can you paint the faces of a regular tetrahedron with four colors if each face is painted a different color? (Assume that two paintings that can be oriented to look the same are considered indistinguishable)

   *There are 4! ways to paint 4 faces with 4 different colors, and there are 4*3 orientations of the tetrahedron, so there are only 4!/4*3 = **2** distinct ways to paint the faces.*

5. A circular table has 60 chairs around it. There are N people seated at this table so that the next person seated must sit next to someone. Find the smallest possible value of N. (AHSME 1991 #15)

   *If every third seat is occupied, filling 20 seats, then every unoccupied seat has a person sitting next to it, so N could be **20**. To see that N must be at least 20, note that any seating which satisfies the conditions cannot contain a gap of more than 2 unoccupied seats between any occupied seats. If we regard adjacent occupied seats as having a gap of 0 between them, every seating of N people contains N gaps, all of which must be less than 2. Thus N+2N = 3N is at least as big as the sum of N and all the which is 60, therefore $3N \geq 60$ and $N \geq 20$.*

   Note that there are simpler arguments that work when there are 60 seats around the table, but this proof has the advantage of working just as well when the *number of seats is not a multiple of 3 (e.g. for 59 seats N must still be at least 20 since $3N \geq 59$, and for 61 seats $3N \geq 61$ implies $N \geq 21$).*

6. How many different sequences of the numbers {0,1,2} of length 10 do not contain any of the subsequences 12, 23, or 31? 3222132111 is such a sequence. (AIME 2003B #3)

   We can construct such a sequence starting with any digit (3 choices). Regardless of which digit we choose, exactly one digit is excluded as a possibility for the second digit, so we have two choices for the second digit, and similarly two choices are excluded for each subsequent digit for a total of $3*2^9$ = ***1536*** *sequences. Note that we can construct any of the desired sequences using our method and that we never construct the same sequence twice.*

   Note that the combinatorial nature of the excluded sequences made this problem much easier than it would have been if only the subsequence "12" had been excluded. This particular problem can be solved using the recurrence F(0) = 1, F(1) = 3, and F(n) = 3F(n-1) – F(n-2). We will learn how to solve problems with recurrences and how to compute the values of recurrences like this one in a later lecture.

7. A decimal number is called "increasing" if each digit is greater than the previous one (e.g. 24589 is one). How many 5 digit increasing numbers are there? (AIME 1992 #2)

   Every "increasing" number has 5 non-zero digits, since the first digit can't be 0 and each subsequent digit must be larger.. There are (9 5) ways of choosing five distinct non-zero digits, and any group of five distinct digits can be put in increasing order in exactly one way. Therefore the answer is (9 5) = ***126****.*

8. Let S = {1, 2, …, 10 }. Find the number of unordered pairs A, B where A and B are disjoint non-empty subsets of S. (counting unordered pairs simply means we don't distinguish the pair A,B and B,A) (AIME 2002B #9)

   We first count the number of ordered pairs of disjoint subsets of S. If we go through the elements of S, for each of elements we can choose to put it in either set A, set B, or neither (but not both), so we are making a sequence of 10 choices with 3 options each time. There are thus 3^{10} different sequences of choices, and each results in a distinct ordered pair of subsets.

   If A and B are distinct, there are exactly two ordered pairs, (A,B) and (B,A), for each unordered pair, otherwise there is just one. Given that A and B are disjoint the only case where they are identical is when they are both empty. Thus there are $(3^{10}-1)/2 + 1$ unordered pairs of disjoint subsets of S.

   Finally, we need to deal with the restriction that A and B must both be non-empty, which we will do by counting the complement, i.e. the number of unordered pairs where one (or both) is empty. If either A or B are empty, the unordered pair A and B will simply consist of the empty set together with some subset of S (note this includes the case where A and B are both empty). Since there are 2^{10} subsets of S, there are exactly 2^{10} unordered pairs of subsets where one (or both) is the empty set.

   Subtracting out these cases results in a total of $(3^{10}-1)/2 + 1 - 2^{10}$ = ***28,501****.*

9. A 7 digit phone number $d_1d_2d_3d_4d_5d_6d_7$ is called memorable if $d_1d_2d_3$ is exactly the same sequence as $d_4d_5d_6$ or $d_5d_6d_7$ (possibly both). (e.g. 4357435 is memorable). Assuming each d_i can be any decimal digit (so d_1 could be 0), how many memorable telephone numbers are there? (AHMSE 1998 #24)

   *There are 10^3*10 phone numbers where $d_1d_2d_3 = d_4d_5d_6$, the same number where $d_1d_2d_3 = d_5d_6d_7$, and exactly 10 phone numbers were $d_1d_2d_3 = d_4d_5d_6 = d_5d_6d_7$ since in this case all the digits must be identical. The total number of memorable telephone numbers is therefore $2*10^3*10 – 10$ = **19,990**.*

10. How many triangles can be formed with vertices on a 4x4 grid of points? (AHSME 1993 #28)

   Every triangle has three vertices which must be among the 16 points in the grid. There are (16 3) ways to choose three points, but the three points chosen must not lie in the same line. The number of subsets of three points which are co-linear can be counted by cases. There are 4 horizontal and 4 vertical lines in the grid. These lines all have 4 points each, so there are (4 3) ways to choose 3 points from each of these lines. There are also 2 diagonal lines with 4 points. In addition there are 4 diagonal lines with 3 points (they lie parallel to and on either side of the 2 main diagonals), and there is only (3 3) = 1 way to choose 3 points on one of these lines. No other lines in the grid contain 3 points.

   Putting it all together we have found (4+4+2)(4 3) + 4*(3 3) distinct ways to pick 3 co-linear points, so there are a total of (16 3) – (10*(4 3) + 4*(3 3)) = **516** triangles.*