Combinatorics: The Fine Art of Counting

Week Five Problems

Before perusing this week's menu, please be sure that you have sampled a broad range of last week's problems and read through the solutions to check your understanding of the concepts. This week's problems build on the same theme – binomial coefficients in all their glory. Please begin with a few appetizers and then move on to the Binomial Buffet for a combinatorial feast.

Appetizers

1. A landscaper is planting a row of 12 trees including 4 birch trees and 8 pine trees. How many ways can this be done without having two birch trees next to each other?

2. How many solutions does the equation $a+b+c+d+e+f = 2006$ have where a, b, c, d, e, and f are all positive integers?

3. In how many ways can 8 identical balls be distributed among 4 distinct bins? In how many ways can 4 identical balls be distributed among 8 distinct bins?

4. A piñata containing 10 Tootsie Rolls and 5 Hershey Kisses breaks open and the candy is eagerly collected by 8 children. In how many ways can the candy be distributed among the children?

5. A drawer contains 8 distinct pairs of gloves each of which consists of a matching left and right glove. If you pull out four individual gloves at random, what is the probability that you have at least one pair of matching gloves? What is the probability you have two pairs of matching gloves?

6. How many non-decreasing sequences of 7 decimal digits are there? (3334559 and 0223448 are two examples).

7. On an integer lattice, how many direct paths are there from the origin (0,0) to the point (n,n)? Now draw a diagonal line from (n,0) to (0,n). Every direct path from (0,0) to (n,n) must cross this line at some point of the form (k,n-k). How many direct paths from (0,0) to (n,n) go through the point (k,n-k)? Now write a combinatorial identity equating your answer to the first question to the sum of your answers to the second question.

Binomial Buffet

The problems below all involve proving combinatorial identities for various summations that may arise in combinatorial problems. To make the problems easier/harder, you are only given the left hand side of the identity and have to determine the right hand side. In each case you should obtain a simple expression involving at most one or two binomial coefficients. For each problem, start by computing the sum for several small values and see if you can form a hypothesis as to what the right hand side might look like. Then try to prove a result either by using a combinatorial argument (e.g. a bijection), applying an identity you already know (e.g. the hockey stick), and/or using algebraic manipulations.

Keep an eye out for simple rearrangements that may make the problem easier, e.g. strategically changing (n k) to (n n-k), or using Gauss's addition trick – compute the sum forward and backwards. When constructing combinatorial arguments, bit-strings and groups of people (e.g. boys and girls) are useful things to count in different ways.

1. **(n 1) + 2*(n 2) + 3*(n 3) + …+ n*(n n) =** ?
   e.g. for n = 3:   (3 1) + 2*(3 2) + 3*(3 3) = 12

2. **(n 0)*(n 1) + (n 1)*(n 2) + (n 2)*(n 3) + … + (n n-1)*(n n) =** ?
   e.g. for n = 3: (3 0)*(3 1) + (3 1)*(3 2) + (3 2)*(3 3) = 15

3. **(n 0) + (n+1 1) + (n+2 2) + … + (n+k k) =** ?
   e.g. for n=3, k=2: (3 0) + (4 1) + (5 2) = 15

4. **(n 0)^2 + (n 1)^2 + (n 2)^2 + … + (n n)^2 =** ?
   e.g. for n=3:   (3 0)^2 + (3 1)^2 + (3 2)^2 + (3 3)^2 = 20

5. **(m 0)*(n k) + (m 1)*(n k-1) + (m 2)*(n k-2) + … + (m k)*(n 0) =** ?
   e.g. for m=4,n=3,k=2: (4 0)*(3 2) + (4 1)*(3 1) + (4 2)*(3 0) = 21

6. **(n k)*(k k) + (n k+1)*(k+1 k) + (n k+2)*(k+2 k) + … + (n n)*(n k) =** ?
   e.g. for n=4, k=2: (4 2)*(2 2) + (4 3)*(3 2) + (4 4)*(4 2) = 24

7. **(n 1) + 3*(n 3) + 5*(n 5) + … =** ?   (sum over odd binomial coefficients)
   e.g. n=3: (3 1) + 3*(3 3) = 4
       n=6: (6 1) + 3*(6 3) + 5*(6 5) = 96

8. **(n 0)*(n k) + (n 1)*(n-1 k-1) + (n 2)*(n-2 k-2) + … + (n k)*(n-k 0) =** ?
   e.g. for n=5,k=2: (5 0)*(5 2) + (5 1)*(4 1) + (5 2)*(3 0) = 40

Multinomial Dessert Menu

9. **What is the coefficient of the term $a^i b^j c^k$ in the expansion of $(a+b+c)^n$?**
   e.g. for $(a+b+c)^4$ the coefficient of $a^1 b^2 c^1$ is 12.

10. **What is the sum of the coefficients of $(a+b+c)^n$?**
    e.g. for $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ac$ the answer is 9.

11. **Generalize your answers to problems 9 and 10 to more than three variables**.