Combinatorics: The Fine Art of Counting

Week Four Problems

Please read through the entire menu and try to classify each problem into one of the following types: Counting Subsets, Distinct Partitions, Block Walking, or Summing (hockey stick). Some problems will cover more than one category, in which case you should try to identify which part of the problem falls into each category. Once you have done this, pick at least one of each type of problem to solve. In addition, everyone should attempt to answer question #1.

A la Carte Menu

1. This problem addresses several different types of counting that we have encountered. You will be asked to compute 4 numbers. Before computing these numbers, first try to determine what order you would intuitively expect the answers to be in, i.e. which should have the biggest answer, which the next biggest, and so on. Then compute the 4 numbers and compare the results with your intuition. If they disagree, figure out which is wrong and why.

   An ice-cream store specializes in super-sized deserts. Their must famous is the "quad-cone" which has 4 scoops of ice-cream stacked one on top of the other. The order of scoops on a cone is distinct. They also offer a "quad-sundae" which has 4 scoops of ice-cream mixed together in a bowl. Once the scoops are in the bowl, you can't distinguish their order, you can only tell how many of each flavor there are. Note however that a sundae with 3 scoops of vanilla and 1 scoop of chocolate is different from a sundae with 3 scoops of chocolate and 1 scoop of vanilla. The store has 10 different flavors of ice cream to choose from.

   1. How many different quad-cones can you get?
   2. How many have four distinct flavors?
   3. How many different quad-sundaes can you get?
   4. How many have four distinct flavors?

2. Suppose Gauss's school teacher had asked him to add up 1 + (1+2) + (1+2+3) + (1+2+3+4) + … + (1+2+3+…+100). What would the answer be? Now suppose Gauss really got into trouble and had to do the same thing for 100 more days in a row, only each day he had to go one sum further, i.e. the last number he added on the next day was 101, and so on until the 100^{th} day when he computed the sum of 1 + (1+2) + (1+2+3) + (1+2+3+4) + … + (1+2+3+…+200). The teacher than asked him to add up all the sums from all 100 days. What was his answer?

3. A donut shop sells donut holes in boxes with 24 donut holes per box. Assuming there are 8 different flavors of donut holes, how many distinct boxes of donut holes could you buy? (Assume you can't distinguish the arrangement of the donut holes inside the box).

Now suppose you look inside the box and notice you didn't get any donut holes in some of the flavors offered and insist that they give a new box with at least one donut hole of each flavor in it. How many distinct boxes of donut hole could they give you? What if you insist on at least two of each flavor?

The donut shop also sells bagels by the half-dozen, with 8 different types to choose from. How many different bags of a half-dozen bagels could you buy?

4. To celebrate the Holiday season, a local radio station decided to play the "Twelve days of Christmas" once on the first day of Christmas, twice on the second day, and so on playing the song 12 times on the 12^{th} day of Christmas. If we added up all the presents mentioned, how many presents would we there be in total?

5. Plain Jane has 5 identical narrow rings that she likes to wear. She can wear them on any of her 8 fingers (but not her thumbs), and they are narrow enough that she can fit all 5 on one finger if she chooses to. How many different ways can Jane wear her rings? (note that Jane's rings may be plain, but she can tell her fingers apart). If she puts at most one ring on each finger, how many ways are there for her to wear her rings?

   Suppose Jane is tired of being plain and paints her rings five different colors so she can tell them apart. How does this change your answers above?

6. Sally lives in a city with a square grid of numbered streets which run east-west and numbered avenues that run north-south. Her house is located on the corner of 0^{th} Street and 0^{th} Avenue. Her aunt lives at the corner of 5^{th} St. and 3^{rd} Ave. How long is the shortest route (along streets or avenues) to her aunt's house? How many direct routes can Sally take to her aunt's house?

   There is a grocery store at the corner of 2^{nd} St. and 2^{nd} Ave. If Sally needs to stop at the store on her way to her Aunt's, how many direct routes to her Aunt's house take her through the intersection of 2^{nd} St. and 2^{nd} Ave?

   At her Aunt's house Sally hears on the radio that there has been an accident at the corner of 1^{st} St. and 2^{nd} Ave. Assuming she avoids this intersection, how many direct routes can she take home?

7. Let S be the set of numbers {1,2,3,…,10}. Let A be a particular subset of S with 4 elements, e.g. {1,2,3,4}. Now let B be any subset of S with 4 elements chosen at random. What is the probability that B = A? What is the probability that A and B are disjoint? What is the probability that A and B overlap?

   Now let B be a subset of S of any size chosen at random. What is the probability that B contains A? What is the probability that A contains B?

8. Consider a tetrahedral stack of balls built by stacking a triangle with 1+2+…+10 balls on the bottom, 1+2+…+9 balls in the next layer, and so on with a single ball on top. How many balls are in this stack?. Now consider a square pyramid of balls constructed by stacking a 10x10 square of balls on the bottom, 9x9 balls in the next layer and so on with a single ball on top. How many balls are in this stack? Express your answer using binomial coefficients and then check it by computing $1^2 + 2^2 + \ldots + 10^2$.

9. Consider the grid of points with integer coordinates in an x-y coordinate system. Now imagine an ant starting at the origin that can crawl up, down, left, or right at each step along a path, i.e. if the ant is at (a,b) it can move to (a-1,b), (a+1,b), (a,b-1), or (a,b+1) in the next step. Define an outward path to be a path that always takes the ant further away from the origin. Label each point with integer coordinates with the number of outward paths that lead from the origin to that point (it may be helpful to draw a picture). Compute the labels on the points (1,2), (-4,0), and (3,-2). Give a general formula for the label on the point (a,b) where a and b are arbitrary integers (possibly negative). Now answer the following two questions:

   How many different points are possible end-points of an n step outward path?

   How many possible outward paths of length n are there in total?

10. A children's stacking game contains a base with a pole in the center, 7 identical discs in each of 7 different sizes (49 discs total) with a hole in the middle of each disc, and a cap which screws on the top of the pole. The base is shaped so that only the biggest disc (size 7) will fit properly. Similarly the cap has a groove that will only accommodate the smallest disc (size 1). The object of the game is to create a complete stack of 7 discs with a size 7 disc on the bottom and a size 1 disc at the top, subject to the constraint that you can never stack a larger disc on top of a smaller one (e.g. 7654321, 7775331, and 7222211 are all complete stacks).

    How many different complete stacks are there?

11. There are 10 positive integers less than 100 whose decimal digits sum to 9. (9, 18, 27, 36, 45, 54, 63, 72, 81, and 90). How many positive integers less than 10,000 have digits that sum to 9? How many have digits that sum to 10? (be careful here – no single digit can have the value 10). How about 15?

12. Consider a convex regular decagon (10-sided polygon) with vertices labeled A, B, C, D, E, F, G, H, I, J. How many different triangles can be formed among these vertices that do not share any edges with the decagon? Two triangles are distinct if they have different sets of vertices, otherwise they are the same.

    This problem can be solved in a couple of different ways, but they all require careful thought. Whatever method you use, check that your method gives the right answer if you apply it to a hexagon – there are exactly two triangles which do not share any edges with a labeled hexagon ({A,C,E} and {B,D,F}).

Once you are satisfied with your solution, try to answer the same question for convex quadrilaterals, i.e. how many different convex quadrilaterals can be formed using the vertices of a labeled decagon that don't share any edges with the decagon? (Note that a set of n vertices that lie on the perimeter of any convex shape determines a unique convex polygon with n sides – there is exactly one convex quadrilateral for each subset of four vertices).

Depending on which method you used to solve the triangle question, this problem will be either easy or quite tricky. As above, make sure that your approach gives the right answer on a labeled octagon where there are exactly two quadrilaterals which satisfy the problem.