Combinatorics: The Fine Art of Counting

Week One Menu

Please start with an appetizer if you wish and then choose one salad and one main course. You are welcome to more if you are extra hungry, and there are a few tasty tidbits for desert for those who still have an appetite.

Appetizer

1. According to legend the ancient Greeks used to play soccer using a regular icosahedron for a ball, until Archimedes came along and suggested that should shave off the corners of the icosahedron to create a truncated icosahedron. This led to the modern soccer ball shape we use today which is semi-regular polyhedron with vertex degree 3 and two hexagons and one pentagon incident to each vertex.

   Compute the number of vertices, edges, and faces of the soccer ball and verify that they satisfy Euler's formula $V + F – E = 2$.

Salads

2. The complete graphs K_1, K_2, K_3, and K_4 are all planar. Prove that K_5 is not planar. What about K_n for $n > 5$?
3. The hypercube graphs H_1, H_2, and H_3 are all planar. Prove that H_4 is not planar. What about H_n for $n > 4$?

Main Courses

4. Given a regular polyhedron with V vertices of degree d, F faces of degree c, and E edges, truncating the polyhedron will result in a semi-regular polyhedron. Let V', F', E', and d' be the vertices, faces, edges and vertex degree of the truncated polyhedron and let c_1' and c_2' be the degrees of the faces of the truncated polyhedron.

   Find simple expressions for V', F', E', d', and c_1' and c_2' in terms of V, F, E, c, and d. Use these expressions to compute the result of truncating each of the five regular polyhedra.
5. An extreme way to truncate a regular polyhedron is to slice off the vertices at a depth which bisects the edges creating a single vertex at the center of each edge, rather than two vertices along each edge as in a normal truncation. The resulting polyhedron will be different from than the one

obtained by the normal truncation process, but it will produce either a semi-regular or in one case regular polyhedron.

As above, find simple expressions for V', F', E', d', and c_1' and c_2' in terms of V, F, E, c, and d. Use these expressions to compute the result of truncating each of the five regular polyhedra in this fashion. How many new semi-regular polyhedra can be obtained in this way?

Dessert

6. All of the proofs of non-planarity we have seen rely on showing that the graph in question has "too many edges" to be planar (either $E > 3V – 6$ or $E > 2V – 4$ for triangle-free graphs). While having a small number of edges is a necessary condition for a graph to be planar, it is not always sufficient.

   Give an example of a non-planar graph where $E <= 2V – 4$

7. There are seven distinct semi-regular polyhedra which can be obtained by truncating or completely truncating the five regular polyhedra as described in problems 4 and 5 above. There are six other convex semi-regular polyhedra which together with these seven make up the thirteen Archimedean solids. There are also two infinite families of convex semi-regular polyhedra which are usually not classified as Archimedean solids.

   How many of these semi-regular polyhedra can you find? (Hint: one of the infinite families is a very familiar class of solid shapes).