# *Combinatorics: The Fine Art of Counting*

## The Final Challenge – Part One Solutions

**Whenever the question asks for a probability, enter your answer as either 0, 1, or the sum of the numerator and denominator of a reduced fraction equal to the probability. For example if the probability is 0.8 = 4/5, the answer would be 9.**

1. A landscaper has 10 bushes that he is going to plant in 4 different sections of a yard. Assuming that he puts at least one bush in each section, how many different ways can he do this?

| 1 point | | | 8 | 4 |
|---|---|---|---|---|

   *(6+3 3) = 84*

2. Call a 7-digit phone number *good* if it contains exactly 4 distinct non-zero digits, does not have a 1 in the first or second position, and both the first 3 digits and the last 4 digits are palindromes. 343-5995, and 929-3113 are examples of good phone numbers. How many good phone numbers are there?

| 1 point | 2 | 3 | 5 | 2 |
|---|---|---|---|---|

   *8*7*7*6 = 2,352*

3. (AHSME 1994) A bag of popcorn contains 2/3 white kernels and 1/3 yellow kernels. Only half the white kernels will pop, whereas 2/3 of the yellow ones will. A random kernel is selected from the bag and it pops when placed in the popper. What is the probability it is white?

| 1 point | | | | 8 |
|---|---|---|---|---|

   *P(W&P) = 1/3, P(P) = (2/3)*(1/2)+(1/3)*(2/3) = 5/9*
   *P(W|P) = P(W&P) / P(P) = 3/5*

4. (AIME 2005B) A game uses a deck of $n$ different cards where $n \geq 6$. The number of possible sets of 6 cards that can be drawn from the deck is 6 times the number of possible sets of 3 cards than can be drawn. Find $n$.

| 1 point | | | 1 | 3 |
|---|---|---|---|---|

   *(n 6) = 6*(n 3) => n = 13.*

5. A space-diagonal connects two vertices of a polyhedron which do not lie on the same face. The truncated icosahedron (a.k.a. a soccer ball) is a semi-regular polyhedra with 12 pentagons and 20 hexagons and vertex degree 3. How many space-diagonals does it have?

| 2 points | 1 | 4 | 4 | 0 |
|---|---|---|---|---|

   *60 vertices, 90 edges, 12*5 + 20*9 = 240 face diagonals*
   *(60 2) – 240 – 90 = 1,440*

6. (AIME 1990) Let *n* be the smallest integer that is a multiple of 75 and has exactly 75 positive integral divisors including 1 and itself. Find n/75?

| 2 points | | 4 | 3 | 2 |
|---|---|---|---|---|

*$2^4*3^4*5^2$ / 75 = 432*

7. (AIME 1990) A fair coin is tossed 10 times. What is the probability that heads never occurs on consecutive tosses?

| 2 points | | | 7 | 3 |
|---|---|---|---|---|

*1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144*
*144 / $2^{10}$ = 9 / 64*

8. (AIME 1998) Consider a set of dominos in which the ends of each domino are labeled with *distinct* integers from 1 to 40 (inclusive) and all possible combinations are represented. A proper sequence of dominos is a line of dominos laid end-to-end where adjacent ends having matching numbers. What is the longest proper sequence that can be formed with this set?

| 2 points | | 7 | 6 | 1 |
|---|---|---|---|---|

*Remove 19 non-adjacent edges from $K_{40}$ to get a graph with an Eulerian trail.*
*(40 2) – 19 = 761*

9. (AIME 2005) How many positive integers have exactly three proper divisors, each of which is less than 50? (A proper divisor of a positive integer n is a positive integer divisor of n other than n itself).

| 2 points | | 1 | 0 | 9 |
|---|---|---|---|---|

*15 primes less than 50 => (15 2) = 105 numbers of the form n=pq*
*4 squares of primes less than 50 => 4 numbers of the form n=$p^3$*

10. (AIME 2004B) A jar has 10 red candies and 10 blue candies. Terry picks 2 candies at random, then Mary picks 2 candies at random. What is the probability they get the same combination of colors, irrespective of order?

| 2 points | | 4 | 4 | 1 |
|---|---|---|---|---|

*(2*10*9*8*7 + 4*10*10*9*9) / 20*19*18*17 = 118 / 323*

# *Combinatorics: The Fine Art of Counting*

## The Final Challenge – Part Two Solutions

**Whenever the question asks for a probability, enter your answer as either 0, 1, or the sum of the numerator and denominator of a reduced fraction equal to the probability. For example if the probability is 0.8 = 4/5, the answer would be 9.**

11. Party favor bags for a party each contain either 3 lollipops which may be any of 5 flavors, or 6 gum-balls which may be any of 3 flavors (but not both). How many different bags can be assembled?

| | | | | |
|---|---|---|---|---|
| **1 point** | | | **6** | **3** |

*$\binom{3+4}{4} + \binom{6+2}{2} = 35 + 28 = 63$*

12. Consider the 3-dimensional space of points with integer coordinates (x,y,z). Any point can be reached from the origin (0,0,0) by taking steps of 1 unit in the positive or negative x, y, or z direction moving from point to point in the grid. A direct path between two points is a path which uses as few steps as possible, e.g. a direct path from the origin to (-3,2,-2) takes 7 steps. How many different direct paths are there from the origin to the point (-3,2,-2)?

| | | | | |
|---|---|---|---|---|
| **1 point** | | **2** | **1** | **0** |

*Mississippi Rule: $7! / (3!*2!*2!) = 210$*

13. In the 3-dimensional grid of points with integer coordinates described in the previous problem, how many different points can be reached from the origin by direct paths of exactly 7 steps?

| | | | | |
|---|---|---|---|---|
| **2 points** | | **1** | **9** | **8** |

*$8*\binom{4+2}{2} = 120$ points with no zeros,*
*$12*\binom{5+1}{1} = 72$ points with one zero, and $6*\binom{6+0}{0} = 6$ points with two zeros*

14. (AHSME 1999) Six points on a circle are given. Four of the chords joining pairs of the six points are selected at random. What is the probability the four chords form a convex quadrilateral?

| | | | | |
|---|---|---|---|---|
| **1 point** | | | **9** | **2** |

*$\binom{6}{4} / \binom{\binom{6}{2}}{4} = \binom{6}{4} / \binom{15}{4} = 1 / 91$*

15. (AIME 2000B) What is the smallest positive integer with six positive odd integer divisors and twelve positive even integer divisors?

| | | | | |
|---|---|---|---|---|
| **2 points** | | **1** | **8** | **0** |

*$2^2*3^2*5 = 180$*

16. (AMC12 2005) Call a number "prime-looking" if it is composite but not divisible by 2, 3, or 5. The three smallest prime-looking numbers are 49, 77, and 91. There are 168 prime numbers less than 1000. How many prime-looking numbers are less than 1000?

| **2 points** | | **1** | **0** | **0** |
|---|---|---|---|---|

*$D_2 = 499$, $D_3 = 333$, $D_5 = 199$, $D_6 = 166$, $D_{10} = 99$, $D_{15} = 66$, $D_{30} = 33$*
*$D_{2or3or5} = 499 + 333 + 199 – 166 – 99 – 66 + 33 = 733$*
*$999 - D_{2or3or5}$ – #(Primes>5) – #1 = 999 – 733 – 165 – 1 = 100*

17. (AIME 2001) A fair die is rolled four times. What is the probability that each of the final three rolls is at least as large as the roll preceding it?

| **2 points** | | | **7** | **9** |
|---|---|---|---|---|

*Count non-decreasing sequences: (5+4 4) / $6^4$ = 7 / 72*

18. There is a 4x4 grid of intersections made up of the intersections of 1st through 4th Street and 1st through 4th Avenue in the center of a city which has been blanketed by snow. All the snow has been cleared by plows except the road sections between each of the intersections in this grid. A single remaining plow has been sent to clear this snow and can begin work at any intersection. How many different times must the plow interrupt work to drive on roads that have already been cleared in order to finish the job? (If the plow can clear all the snow in one continuous route, your answer should be 0). Note that after finishing one pass of plowing the plow can restart plowing at any intersection.

| **2 points** | | | | **3** |
|---|---|---|---|---|

*The 4x4 grid has 8 intersections with odd degree. Each plow route eliminates at most 2 of these, so at least 4 routes are required => 3 interruptions.*

19. (AIME 1989) When a certain biased coin is flipped 5 times the probability of getting exactly one head is the same as that of getting two heads. What is the probability of getting exactly three heads?

| **2 points** | | **2** | **8** | **3** |
|---|---|---|---|---|

*$(5\ 1)*p*q^4 = (5\ 2)*p^2*q^3$ => $5q = 10p$ => $p = 1/3$.*
*$(5\ 3)*p^3*q^2 = 10*(1/3)^3*(2/3)^2 = 40 / 243$*

20. (AIME 1996) In a five-team tournament, each team plays one game with every other team. Each team has a 50% chance of winning any game it plays (there are no ties). What is the probability that the tournament produces neither an undefeated team nor a winless team?

| **2 points** | | | **4** | **9** |
|---|---|---|---|---|

*$1 – (5*2^6 + 5*2^6 – 5*4*2^3) / 2^{10} = 17 / 32$*

# *Combinatorics: The Fine Art of Counting*

## The Final Challenge – Part Three Solutions

**Whenever the question asks for a probability, enter your answer as either 0, 1, or the sum of the numerator and denominator of a reduced fraction equal to the probability. For example if the probability is 0.8 = 4/5, the answer would be 9.**

21. A grocery store display consists of a tetrahedral stack of cans of tomato soup. There is one can on top and each layer below is an equilateral triangle of cans which has one more can per side than the layer above it. If there are 17 cans per side in the bottom layer, how many cans are in the stack in total?

| | | | | |
|---|---|---|---|---|
| **1 point** | | **9** | **6** | **9** |

*(17+2 3) = 969*

22. How many distinct ways can 4 women and 4 men be seated around a circular table so that the genders alternate? (seating arrangements which are circular rotations of each other should not be regarded as distinct).

| | | | | |
|---|---|---|---|---|
| **1 point** | | **1** | **4** | **4** |

*3!*4! = 144*

23. How many distinct ways can 4 men and 4 women who are married couples be seated around a circular table so that genders alternate and no spouses are adjacent? (seating arrangements which are circular rotations of each other should not be regarded as distinct).

| | | | | |
|---|---|---|---|---|
| **1 point** | | | **1** | **2** |

*Seat a man, the women can be seated in 2*3*2 = 12 ways, at which point the seating of the rest of the men is determined.*

24. (AHSME 1997) Two fair six-sided dice are modified so that one has the 3 replaced by a 4 (so it has two 3's) and one has the 4 replaced by a 3 (so it has two 4's). When these two dice are rolled, what is the probability that the sum is an odd number?

| | | | | |
|---|---|---|---|---|
| **1 point** | | | **1** | **4** |

*(2/3)*(2/3) + (1/3)*(1/3) = 5 / 9*

25. (AIME 2005B) Find the number of positive integers that are divisors of at least one of $10^{10}$, $15^{7}$, and $18^{11}$.

| | | | | |
|---|---|---|---|---|
| **2 points** | | **4** | **3** | **5** |

*$2^{10}*5^{10}$ has 11*11=121, $3^{7}*5^{7}$ has 8*8=64, $2^{11}*3^{22}$ has 12*23=276*
*$5^{7}$ has 8, $2^{10}$ has 11, $3^{7}$ has 8, and they all have the divisor 1 in common*
*121 + 64 + 276 – 8 – 11 – 8 + 1 = 435*

26. (AIME 2001B) Club Truncator is in a soccer league with six other teams, each of which it plays once. In any of its 6 matches, the probabilities that Club Truncator will win, lose, or tie are each 1/3. What is the probability that Club Truncator finishes the season with more wins than losses?

| | | | | |
|---|---|---|---|---|
| **3 points** | | **3** | **4** | **1** |

*P(Winning Season) = P(Losing Season) = (1 – P(.500 Season)) / 2*
*P(.500 Season) = [(6 3) + (6 2,2,2) + (6 4,1,1) + (6 6)] / $3^6$ = 47 / 243*
*P(Winning Season) = (1 – 47/243) / 2 = 98 / 243*

27. The cuboctahedron is a semi-regular polyhedron which can be constructed by placing a vertex at the mid-point of each edge of the cube and connecting vertices which lie on the mid-points of edges adjacent in the cube. It can also be constructed via the same process starting with an octahedron. Compute the number of faces this polyhedron has and determine the minimum number of colors required to color the faces so that no two faces with a common edge have the same color. Multiply this number times the number of faces and enter the result.

| | | | | |
|---|---|---|---|---|
| **2 points** | | | **2** | **8** |

*The cuboctahedron has 8 triangular faces and 6 square faces, with every vertex degree 4. The dual of the cuboctahedron contains no odd cycles (every face is a square), so its vertices can be properly 2-colored, which means the faces of the cuboctahedron can be 2-colored. 2*14 = 28.*

28. (AIME 2003B) A bug starts at a vertex on an equilateral triangle. On each move it randomly selects one of two vertices where it is not currently located and moves to that vertex. What is the probability that after 10 moves the bug is back where it started?

| | | | | |
|---|---|---|---|---|
| **3 points** | | **6** | **8** | **3** |

*Count sequences of +1s and -1s which sum to 0 mod 3 (i.e. 0, ±3, ±6, or± 9).*
*[(10 5) + 0 + 2*(10 8) + 0] / $2^{10}$ = (252 + 90) / 1024 = 171 / 512*

29. A caterpillar graph is a connected graph with no cycles (i.e. a tree) in which the internal vertices all lie on a path (called the spine). Call a caterpillar “leggy” if every internal vertex has degree > 2. How many different (non-isomorphic) leggy caterpillars are there with 10 vertices.

| | | | | |
|---|---|---|---|---|
| **2 points** | | | | **9** |

*There is 1 with 1 internal vertex, 3 with 2, 4 with 3, and 1 with 4.*

30. (AIME 1993) Alfred and Bonnie play a game in which they take turns tossing a fair coin. The winner is the first person to obtain a head. They play this game several times, with the stipulation that the loser of a game goes first in the next game. Suppose Alfred goes first in the first game, what is the probability that he wins the $6^{th}$ game?

| | | | | |
|---|---|---|---|---|
| **4 points** | **1** | **0** | **9** | **3** |

$P_k$ = *probability Alfred wins* $k^{th}$ *game.* $P_1 = ½ + (1/2)^3 + (1/2)^5 + … = 2/3$
$P_k = P_{k-1}*1/3 + (1- P_{k-1})*2/3 = 2/3 – 1/3P_{k-1}$
$P_2 = 4/9$, $P_3 = 13/27$, $P_4 = 40/81$, $P_5 = 122/243$, $P_6 = 364/729$