PI. PROPERTIES OF INTEGRALS

For ease in using the definite integral, it is important to know its properties. Your book lists the following[1] (on the right, we give a name to the property):

(1) $$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$ **integrating backwards**

(2) $$\int_a^a f(x)\,dx = 0$$

(3) $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$ **interval addition**

(4) $$\int_a^b (f+g)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$ **linearity**

$$\int_a^b C\,f(x)\,dx = C\int_a^b f(x)\,dx$$ **linearity**

(5) $$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx \quad \text{if } f(x) \le g(x) \text{ on } [a,b]$$ **estimation**

Property (5) is useful in estimating definite integrals that cannot be calculated exactly.

Example 1. Show that $\int_0^1 \sqrt{1+x^3}\,dx < 1.3$.

Solution. We estimate the integrand, and then use (6). We have

$$x^3 \le x \quad \text{on } [0,1];$$

$$\int_0^1 \sqrt{1+x^3}\,dx \le \int_0^1 \sqrt{1+x}\,dx = \frac{2}{3}(1+x)^{3/2}\Big]_0^1 = \frac{2}{3}(2\sqrt{2}-1) \approx 1.22 < 1.3\,.$$

We add two more properties to the above list.

(6) $$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx\,.$$ **absolute value property**

Property (6) is used to estimate the size of an integral whose integrand is both positive and negative (which often makes the direct use of (5) awkward). The idea behind (6) is that on the left side, the intervals on which $f(x)$ is negative give a negative value to the integral, and these "negative" areas lower the overall value of the integral; on the right the integrand has been changed so that it is always positive, which makes the integral larger.

Example 2. Estimate the size of $\int_0^{100} e^{-x}\sin x\,dx$.

[1]see Simmons pp. 214-215

Solution. A crude estimate would be

$$\left|\int_0^{100} e^{-x}\sin x\,dx\right| \le \int_0^{100} e^{-x}|\sin x|\,dx$$
$$\le \int_0^{100} e^{-x}\,dx, \qquad \text{by (5), since } |\sin x| \le 1;$$
$$= -e^{-x}\Big]_0^{100} = -e^{-100}+1 < 1.$$

A final property tells one how to change the variable in a definite integral. The formula is the most important reason for including dx in the notation for the definite integral, that is, writing $\int_a^b f(x)\,dx$ for the integral, rather than simply $\int_a^b f(x)$, as some authors do.

$$(7) \quad \int_c^d f(u)\,du = \int_a^b f(u(x))\,\frac{du}{dx}\,dx, \quad \begin{cases} u = u(x), \\ c = u(a), \\ d = u(b). \end{cases} \qquad \textbf{change of variables formula}$$

In words, we can change the variable from u to x, provided we

(i) express du in terms of dx; (ii) change the limits of integration.[2]

There are various possible hypotheses on $u(x)$; the simplest is that it should be differentiable, and either increasing or decreasing on the x-interval $[a, b]$.

Example 3. Evaluate $\displaystyle\int_0^1 \frac{du}{(1+u^2)^{3/2}}$ by substituting $u = \tan x$.

Solution. For the limits, we have $u = 0, 1$ corresponding to $x = 0, \pi/4$; $\tan x$ is increasing.

$$\int_0^1 \frac{du}{(1+u^2)^{3/2}} = \int_0^{\pi/4} \frac{\sec^2 x}{\sec^3 x}\,dx$$
$$= \int_0^{\pi/4} \cos x\,dx = \sin x\Big]_0^{\pi/4} = \frac{\sqrt{2}}{2}.$$

Proof of (7). We use the First Fundamental Theorem[3] and the chain rule. Let $F(u)$ be an antiderivative:

$$(8) \qquad F(u) = \int f(u)\,du;$$
$$\frac{d}{dx}F(u(x)) = \frac{dF}{du}\cdot\frac{du}{dx} = f(u)\,\frac{du}{dx}, \qquad \text{by the chain rule. So}$$
$$(9) \qquad F(u(x)) = \int f(u(x))\,\frac{du}{dx}\,dx. \qquad \text{Therefore}$$
$$\int_c^d f(u)\,du = F(d) - F(c), \qquad \text{by the First Fundamental Theorem and (8);}$$
$$= F(u(b)) - F(u(a)) = \int_a^b f(u(x))\,\frac{du}{dx}\,dx,$$

by the First Fundamental Theorem and (9). □

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[2] see Simmons p.339 for a discussion and an example
[3] see the next page