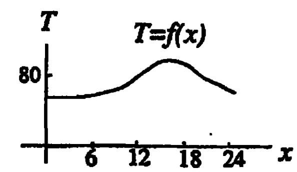
AV. AVERAGE VALUE

What was the average temperature on July 4 in Boston?



The temperature is a continuous function $f(x)$, whose graph over the 24-hour period might look as shown. How should we define the average value of such a function over the time interval $[0,24]$ — measuring time x in hours, with $x=0$ at 12:00AM ?

We could observe the temperature in the middle of every hour, that is, at the times $x_1 = .5, x_2 = 1.5, \ldots, x_{24} = 23.5$, then average these 24 observations, getting

$$\frac{1}{24}\sum_{i=1}^{24} f(x_i)\,.$$

To get a more accurate answer, we could average measurements made more frequently, say every ten minutes.

For a general interval $[a,b]$ and function $f(x)$, the analogous procedure would be to divide up the interval into n equal parts, each of length

(1) $$\Delta x = \frac{b-a}{n}$$

and average the values of the function $f(x)$ at a succession of points x_i, where x_i lies in the i-th interval. Then we ought to have

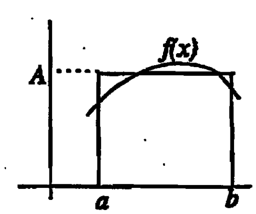
(2) $$\text{average of } f(x) \text{ over } [a,b] \approx \frac{1}{n}\sum_{1}^{n} f(x_i)\,.$$

We can relate the sum on the right to a definite integral: using (1), (2) becomes

(3) $$\text{average of } f(x) \text{ over } [a,b] \approx \frac{1}{b-a}\sum_{1}^{n} f(x_i)\Delta x\,.$$

As $n \to \infty$, the sum on the right-hand side of (3) approaches the definite integral of $f(x)$ over $[a,b]$, and we therefore define the **average value of the function** $f(x)$ on $[a,b]$ by

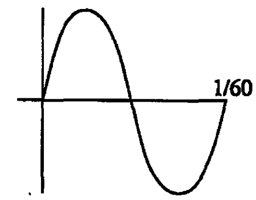
(4) $$A = \text{average of } f(x) \text{ over } [a,b] = \frac{1}{b-a}\int_a^b f(x)\,dx\,.$$



Geometrically, the average value A can be thought of as the height of that constant function A which has the same area over $[a,b]$ as $f(x)$ does. This is so since (4) shows that

$$A\cdot(b-a) = \int_a^b f(x)\,dx\,.$$

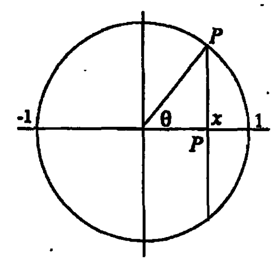
Example 1. In alternating current, voltage is represented by a sine wave with a frequency of 60 cycles/second, and a peak of 120 volts. What is the average voltage?



Solution. The voltage function has frequency $\dfrac{2\pi}{\text{period}} = \dfrac{2\pi}{1/60} = 120\pi$, and amplitude 120, so it is given by $V(t) = 120\sin(120\pi t)$. Thus by (4),

$$\text{average } V(t) = 120\int_0^{1/120} V(t)\,dt = -\frac{120}{\pi}\cos(120\pi t)\Big]_0^{1/120} = \frac{2}{\pi}\cdot 120\,.$$

(We integrate over $[0, 1/120]$ rather than $[0, 1/60]$ since we don't want zero as the average.)



Example 2.

a) A point is chosen at random on the x-axis between -1 and 1; call it P. What is the average length of the vertical chord to the unit circle passing through P?

b) Same question, but now the point P is chosen at random on the circumference.

Solution. a) If P is at x, the chord has length $2\sqrt{1-x^2}$, so we get

$$\text{average of } 2\sqrt{1-x^2} \text{ over } [-1,1] = \frac{1}{2}\int_{-1}^{1} 2\sqrt{1-x^2}\,dx = \text{area of semicircle} = \frac{\pi}{2} \approx 1.6\,.$$

b) By symmetry, we can suppose P is on the upper semicircle. If P is at the angle θ, the chord has length $2\sin\theta$, so this time we get

$$\text{average of } 2\sin\theta \text{ over } [0,\pi] = \frac{1}{\pi}\int_0^{\pi} 2\sin\theta\,d\theta = -\frac{2}{\pi}\cos\theta\Big]_0^{\pi} = \frac{4}{\pi} \approx 1.3\,.$$

(Intuitively, can you see why the average in part (b) should be less than the average in part (a) — could you have predicted this would be so?)

Exercises: Section 4D