In 1974, Erno Rubik created the Rubik's Cube. It is the most popular puzzle worldwide. But now that it has been solved in 7.08 seconds, it seems that the world is in need of a new challenge. Melinda Green, Don Hatch, and Jay Birkenbilt took it upon themselves to meet this challenge. They created a program that is a 4-dimensional analog of the Rubik's Cube. This paper will discuss the properties of the hypercube and the general idea behind the solution to the 4-dimensional Rubik's cube.

First, we must discuss what the fourth dimension is. Most people regard the fourth dimension as time, but that is certainly not how it is going to be presented in this context. We cannot actually perceive a fourth spatial dimension directly, but we can do so by analogy. Suppose that there is a two-dimensional world called Flatland. In Flatland, the world is a plane and all of the inhabitants are two-dimensional creatures. Everything in Flatland exists in the plane of Flatland (that is, not sticking out). Now suppose that you tell some person on Flatland that there exists a third dimension. The person would respond, "Third dimension?! There is no third dimension!" You then proceed to put a rod through the plane of Flatland and say, "See? Look, I told you that there is a third dimension." The person would respond, "No, there isn't, all I see is a circle." The person in Flatland cannot perceive the third dimension, but that does not mean that it does not exist. Similarly, we cannot perceive the fourth spatial dimension, but that does not mean that it does not exist. In the same vein of a two-dimensional world, think about a sheet of paper. For all practical purposes, the sheet of paper has only two dimensions: length and width. But there is of course a third dimension! The third dimension is the thickness of the paper. However, the thickness is so small compared to the other two dimensions that it can be ignored. Similarly, perhaps the fourth dimension is analogous to the paper example. That is, that there is a fourth dimension, but it is so thin (perhaps infinitesimally thin) that it is ignored, or, we just do not notice it. Thus, the fourth dimension is usually defined via analogies to three-dimensional space.

Now that we have an idea of what discussed what the fourth dimension is, we can discuss some properties of the four-dimensional Rubik's cube. Since, as we have pointed out, we cannot have a literal four-dimensional cube to hold and play with, it must be simulated on a computer. However, there are some properties that are necessary of the four dimensional analogue that do not require it being on a computer to accurately describe them. Since the fourth-dimension is defined by analogy to the three-dimensions, the four-dimensional Rubik's cube will be defined by analogy to the three-dimensional Rubik's cube.

On the three-dimensional cube, there are three-dimensional edge pieces and corner pieces. The edge pieces have 2 two-dimensional stickers on them of different colors, while the corner pieces have 3 two-dimensional stickers on them of different colors. In the four-dimensional cube, the edge pieces and corner pieces are four-dimensional and they have three-dimensional stickers on them. The edge pieces have 3 three-dimensional stickers on them, while the corner pieces have 4 three-dimensional stickers on them.

On the three-dimensional cube, there are 6 center pieces: one on the positive and negative ends of the x-axis, y-axis, and z-axis. Well, in the fourth dimension, there is an additional axis, which has a positive and negative end; therefore the four-dimensional cube will have 8 center pieces. The analogies for the edge pieces and corner pieces are a bit harder to describe. Because this is a four-dimensional cube, it introduces some new properties.

There are now “face” pieces and “edge” pieces. In a three-dimensional cube, each face has a 3-by-3 grid of two-dimensional stickers. Well, by analogy for the fourth dimension, there is a 3-by-3-by-3 cube of three-dimensional stickers. On the three-dimensional cube, there is no need to describe what connects the centers. There are no pieces that connect the centers that affect the solved configuration of the cube. However, for the four-dimensional cube, there are pieces between the centers that affect how the cube will be solved. On the three-dimensional cube, the 3-by-3 grid can be thought of as 8 stickers that surround the center. By analogy, the 3-by-3-by-3 cubic face can be thought of as 26 three-dimensional stickers that surround a center. This brings in a new level of pieces, namely the pieces that connect all of the centers together.

The “face” pieces are somewhat analogous to edge pieces on a Rubik’s Cube. These face pieces have only two colors on them. There are a total of 24 of them. There are 6 that connect the central face to the adjacent 6 faces. But that is only on one end of the fourth dimensional axis. On the other end of the fourth-dimensional axis, there are 6 more. Then, there are four face pieces that connect the top face to four of the faces adjacent to the central face. Then there are four face pieces that connect the bottom face to four of the faces adjacent to the central face. Finally, there are the four face pieces that connect those four adjacent faces together. Thus the total is 6+6+4+4+4, which equals 24.

There are also edge pieces. The edge pieces on a three-dimensional cube connect one color of a 3-by-3 grid to an adjacent 3-by-3 grid of another color. By analogy, there are edge pieces on the four-dimensional Rubik’s cube that connect adjacent 3-by-3-by-3 grids together. Since for any face of the four-dimensional cube, there are three adjacent faces of a different color, the edge pieces on the four-dimensional Rubik's cube contain 3 colors. There are a total of 32 of these pieces. There are 12 edge pieces that are connected to one end of the fourth-dimensional axis: 4 in the xy-plane, 4 in the yz-plane, and 4 in the xz-plane. Similarly, there are 12 on the other end of the fourth-dimensional axis. Now, there are 8 more edge pieces. These pieces connect the pieces that are adjacent to each other, but not to the central face, whether that central cube is on the positive or negative axis of the fourth dimension. Thus the total is 12+12+8, which equals 32.

Finally there are the corner pieces. A corner piece of the three-dimensional cube connects one 3-by-3 grid to two adjacent 3-by-3 grids. By analogy, the corners of the four-dimensional Rubik's cube connect three adjacent 3-by-3-by-3 faces together. There is a total of 16 corner pieces. There are 8 corners that connect the central face to the three adjacent 3-by-3-by-3 cubic grids. That is only on one end of the fourth-dimensional axis. There are eight more on the other end. Thus, the total number of corner pieces is 8+8, which equals 16.

Therefore, the total number of pieces on the four-dimensional Rubik's Cube is: 8 center pieces plus 24 face pieces plus 32 edge pieces plus 16 corner pieces, which equals 80 pieces. However, the center pieces never move with respect to each other, so there are actually only 72 moveable pieces.

Now that we have a grasp on the structure of the four-dimensional Rubik's cube, let us now consider what happens when you perform twists on a four-dimensional Rubik's cube. On a three-dimensional Rubik's cube, it is intuitive to think of twisting as rotating about an axis. However, that is not helpful to the understanding of twists on the four-dimensional Rubik's Cube (indeed, any higher dimensional cube), because it is unclear what the fourth dimension is. Thus to say that twisting on the four-dimensional cube is like twisting about a four-dimensional axis is meaningless in understanding how four-dimensional twists work. Therefore, twisting needs to be thought of differently.

The fact that twisting on a three-dimensional cube can be described as rotating about an axis is merely a special case. It does not fully describe the higher order cases. Another way to think about twisting a face on a three-dimensional cube is to say that you pick a face, remove that face from the cube, then rotate it without flipping it over, then put it back onto the cube. There are only 4 ways to do this, and they all involve rotating about the axis perpendicular to the center. Now this concept can be applied to the four-dimensional cube by analogy. To twist a face on a four-dimensional cube, you pick a face, remove that face from the hypercube, rotate it without turning it inside out, and then put it back in place. On the three-dimensional cube, the face was a 3-by-3 grid, so removing it was like taking it off a plane and later putting it back on the plane. However, the four-dimensional cube has a face that is a 3-by-3-by-3 cube, so removing that face is like taking it out of a box, and then later putting it back in the box. Once you take it out of the box, you rotate it. There are a total of 24 ways to do this. You could rotate it 4 ways about the axis in the xy-plane, 4 in the yz-plane, and 4 in the xz-plane. Also, you can rotate about the diagonals of the cube. There are only 3 ways you can rotate about a diagonal. There are 4 diagonals, each connecting opposite corners of the cube that do not contain the same plane. Thus the total number of ways to rearrange a face after a twist is 4+4+4+3+3+3+3, which equals 24.

Now that we have a conceptual understanding of what the fourth-dimension is and what the fourth dimensional Rubik's cube structure will contain, we must now consider the computer program that accurately portrays the four-dimensional Rubik's cube, namely, MagicCube 4D. This applet can be downloaded for free from http://www.superliminal.com/cube/applet.html

The cube seems to look somewhat distorted. This is because of the perspective changes that come from projecting down to a lower dimension. Think about drawing a cube on a sheet of paper. A cube, by definition has all of the sides perpendicular to each other. However, when you draw it on paper, some of the sides may look perpendicular, while some clearly do not look perpendicular. Additionally, in the Flatland example, if you were to try to show an inhabitant of Flatland that the third dimension exists by drawing them a cube on a sheet of paper, that person would only comprehend a group of lines, not the cube itself. A very similar thing happens for the four-dimensional Rubik's cube, except worse. The four-dimensional cube is projected into our three-dimensional world, which is then projected onto a two-dimensional computer screen, so it looks even more distorted than usual. Furthermore, since the four-dimensional cube is projected onto our three-dimensional world, we can never get an understanding of what the four-dimensional cube actually looks like. All we see is a group of cubes. The four-dimensional cube looks essentially like a cube of cubes, but only the center face only looks like a two-dimensional projection of a cube that we are used to seeing. This is because of the distortion that occurs from projecting down two dimensions. Just as drawing a cube on paper can have faces that are square with other distorted faces, putting a four-dimensional object on the screen results in one face looking like a conventionally distorted cube, with the other faces looking like especially distorted cubes.

Earlier, we said that there are eight center pieces. However, on the applet we only see seven center pieces. Again, this can be explained by analogy to the three-dimensional cube. On a three-dimensional cube, it is impossible to see all 6 faces at the same time. Similarly, on a four-dimensional cube, it is impossible to see all 8 faces at the same time. The way the applet is set up, the maximum amount of faces you can see at the same time is seven. However, scrambling the cube inherently involves messing up the eighth face, so in order to solve the four-dimensional Rubik's Cube, you must somehow be able to see the eighth face. How is this possible? Well, on a three-dimensional cube, although all 6 faces cannot be seen at the same time, you can still look at any desired face via a rotation about a principal axis of the cube. The same concept applies to the four-dimensional cube, but now, since it is four-dimensional, there is a new axis to rotate about. In order to see the eighth face, you must rotate about the fourth-dimensional axis. In the applet, you do this by holding down the option key and clicking on a sticker on a face that you want to move to the central core. The eighth face then replaces the face that you just clicked, and the last face that was on the path of moving faces moves out to be the hidden eighth face. It is important to note that this creates a motion that turns the four-dimensional cube inside out. This is what is commonly accepted as a four-dimensional rotation. Also, this four-dimensional rotation does not affect the state of the cube in terms of how solved it is. This makes sense because on a three-dimensional cube, rotating it does not get it any closer to being solved; it just changes the angle at which you view the cube. Similarly, on the four-dimensional cube, the four-dimensional rotation just changes the angle (in the fourth dimension) with which you view the hypercube; it does not get it any closer to being solved.

Now, we must consider a very important part of manipulating the four-dimensional Rubik's cube, namely how a twist is performed on the applet. Recall that a face is a group of 26 stickers that surround a center. When you want to perform a twist, you highlight a sticker and click on it. The left click twists it counterclockwise with respect to the axis of rotation that is being viewed from above. The right click twists it clockwise. The axis of twist goes through the highlighted sticker and the center of the face that contains the highlighted sticker.

It is important to note that clicking on a sticker on a face twists the face into a new position without changing it. When you use the applet, however, it seems that some 3-by-3 slices of one face seem to spin or fly onto another face. This can be explained by the fact that twisting creates the scrambling effect of the four-dimensional Rubik's Cube. On the three-dimensional cube, twisting a face of the cube does not affect the stickers in the sense that all of the stickers stay on the same piece. The state of the stickers on the face does not change, but the effect of the state of the adjacent faces does change. That is how the cube gets scrambled.

An important conceptual leap that needs to be made in order to understand the applet is that the faces and stickers are separated by gaps. There seems to be no physical connection between the stickers, so it seems that any sticker can move anywhere, however, this is not the case. In the applet, although there does not appear to be any connection between the pieces, they are indeed connected! In order to solve the cube, you must keep in mind that the three-dimensional stickers are connected by four-dimensional pieces. However, we cannot see these pieces because that would require that we be in the fourth-dimension. Furthermore, on a real four-dimensional Rubik's cube, all of the faces and stickers would be slammed together. The applet simply presents an exploded view of the real four-dimensional Rubik's cube so that we can see the internal state, or the central face.

By now, we should have a good understanding of the structure of the four-dimensional cube and how the four-dimensional cube works. Now let us discuss how to solve the cube from a scrambled state. In order to solve the cube, it must first be scrambled. To truly solve the puzzle, you must first select from "Full" from the "Scramble" drop-down menu. It is a truly difficult job to solve the cube from a scrambled state. At the time that this paper is written, there have only been 104 people worldwide who have solved the four-dimensional analog of the three-dimensional 3-by-3-by-3 Rubik's Cube.

If the challenge of truly solving a fully scrambled cube seems too daunting, then the applet contains several levels of how scrambled the four-dimensional cube can be. It is much easier to solve the cube from one random twist. Then, once that feels comfortable, you can move on to two random twists then three, then more. Indeed, this can be quite helpful in solving the puzzle. In the three-dimensional puzzle, before diving right into solving the puzzle, it is important to get a good *feel* of the puzzle. It is important to gain the conceptual understanding of moving three-dimensional pieces that contain two-dimensional stickers. Your goal is not to move all of the stickers in the right place. Your goal is to move all of the right pieces in the right place. The same concept is extremely important in solving the four-dimensional puzzle. You must focus on putting the four-dimensional pieces in the right place, not to put the three-dimensional stickers in the right place.

The four-dimensional Rubik's cube was created by an analogy to the three-dimensional Rubik's cube, so it only makes sense that in order to solve the four-dimensional Rubik's Cube, it is at least extremely helpful in understanding how to solve the three-dimensional Rubik's Cube first. On the three-dimensional cube, one way to solve the cube is to first place the two-colored edge pieces in the right place, then the three-colored corner pieces. This essentially means that you are solving from the centers outward. Well, on the four-dimensional puzzle, in order to work from the centers outward, you must solve the two-colored face pieces, then the three-colored edge pieces, and then the four-colored corner pieces. Not surprisingly, solving the two-colored face pieces on the four-dimensional cube is very similar to solving the two-colored edge pieces on the three dimensional cube. Furthermore, solving the three-colored edge pieces on the four-dimensional cube is very similar to solving the three-colored corner pieces on the three-dimensional cube. Thus, it is quite possible to get two-thirds of the way through solving it without learning a lot of new information. Solving the four-colored corner pieces however, requires learning some new sequences, as there is no analog for the four-colored corner pieces on the three-dimensional cube.

The first step in solving the three-dimensional cube is to solve a cross with the two-colored edge pieces. Well, on the four-dimensional cube, the first step is to solve a cross with the two-colored face pieces. The cross on the three-dimensional cube consists of a center piece that is surrounded by 4 edge pieces on the same face. The cross on the four-dimensional cube consists of a center piece that is surrounded by 6 face pieces on the same face. This step consists of solving center pieces in the central face. The next step is to move the outer, hidden face to the center and solve for another cross. After that, the "top face" face pieces need to be solved. This solves all of the face pieces. Next the three-colored edge pieces will be solved. This step depends more heavily on algorithms than on a conceptual understanding of shifting pieces around, much like solving the three-dimensional cube at this point. Finally, the four-colored corner pieces need to be put in the right place, and then the cube is solved.

Sources

http://www.superliminal.com/cube/faq.html

http://www.superliminal.com/cube/solution/solution.htm

http://www.superliminal.com/cube/solution/pages/cube.htm

SP.268 / ESG.SP268 The Mathematics in Toys and Games
Spring 2010