SP.268	Lecture 3	Feb 16, 2010

http://erikdemaine.org/papers/AlgGameTheory_GONC3

Playing Games with Algorithms:

- most games are hard to play well:
- Chess is EXPTIME-complete:
  - $n \times n$ board, arbitrary position
  - need exponential (c^n) time to find a winning move (if there is one)
  - also: as hard as all games (problems) that need exponential time

- Checkers is EXPTIME-complete:

  ⇒ Chess & Checkers are the "same" computationally: solving one solves the other

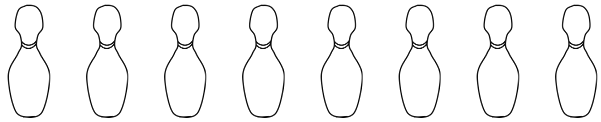
  (PSPACE-complete if draw after poly. moves)

- Shogi (Japanese chess) is EXPTIME-complete
- Japanese Go is EXPTIME-complete
  - U. S. Go might be harder

- Othello is PSPACE-complete:
  - conjecture requires exponential time, but not sure (implied by P $\neq$ NP)

- can solve some games fast: in "polynomial time" (mostly 1D)

Kayles:



[Dudeney 1908]

(n bowling pins)

- move = hit one or two adjacent pins
- last player to move wins (normal play)

Let's play!

First-player win: SYMMETRY STRATEGY

- move to split into two equal halves (1 pin if odd, 2 if even)
- whatever opponent does, do same in other half
  ($K_n + K_n = 0 \ldots$ just like Nim)

Impartial game, so Sprague-Grundy Theory says Kayles $\equiv$ Nim somehow

$$\begin{aligned}
-\ \text{followers}(K_n) &= \{K_i + K_{n-i-1}, K_i + K_{n-i-2} \mid i = 0, 1, \ldots, n-2\} \\
\Rightarrow \text{nimber}(K_n) &= \text{mex}\{\text{nimber}(K_i + K_{n-i-1}), \\
&\qquad \text{nimber}(K_i + K_{n-i-2}) \\
&\qquad \mid i = 0, 1, \ldots, n-2\} \\
-\ \text{nimber}(x+y) &= \text{nimber}(x) \oplus \text{nimber}(y) \\
\Rightarrow \text{nimber}(K_n) &= \text{mex}\{\text{nimber}(K_i) \oplus \text{nimber}(K_{n-i-1}), \\
&\qquad \text{nimber}(K_i) \oplus \text{nimber}(K_{n-i-2}) \\
&\qquad \mid i = 0, 1, \ldots n-2\}
\end{aligned}$$

RECURRENCE! — write what you want in terms of smaller things

How do we compute it?

$\text{nimber}(K_0) = 0$ (BASE CASE)

$$\begin{aligned}
\text{nimber}(K_1) &= \text{mex}\{\text{nimber}(K_0) \oplus \text{nimber}(K_0)\} \\
&\qquad\qquad 0 \oplus 0 = 0 \\
&= 1
\end{aligned}$$

$$\begin{aligned}
\text{nimber}(K_2) &= \text{mex}\{\text{nimber}(K_0) \oplus \text{nimber}(K_1), \\
&\qquad\qquad 0 \oplus 1 = 1 \\
&\qquad \text{nimber}(K_0) \oplus \text{nimber}(K_0)\} \\
&\qquad\qquad 0 \oplus 0 = 0 \\
&= 2
\end{aligned}$$

so e.g. $K_2 + *2 = 0 \Rightarrow$ 2nd player win

$$\begin{aligned}
\text{nimber}(K_3) &= \text{mex}\{\text{nimber}(K_0) \oplus \text{nimber}(K_2), \\
&\qquad\qquad 0 \oplus 2 = 2 \\
&\qquad \text{nimber}(K_0) \oplus \text{nimber}(K_1), \\
&\qquad\qquad 0 \oplus 1 = 1 \\
&\qquad \text{nimber}(K_1) \oplus \text{nimber}(K_1)\} \\
&\qquad\qquad 1 \oplus 1 = 0 \\
&= 3
\end{aligned}$$

$$
\begin{aligned}
\text{nimber}(K_4) &= \text{mex}\{\text{nimber}(K_0) \oplus \text{nimber}(K_3), \\
&\qquad 0 \oplus 3 = 3 \\
&\qquad \text{nimber}(K_0) \oplus \text{nimber}(K_2), \\
&\qquad 0 \oplus 2 = 2 \\
&\qquad \text{nimber}(K_1) \oplus \text{nimber}(K_2), \\
&\qquad 1 \oplus 2 = 3 \\
&\qquad \text{nimber}(K_1) \oplus \text{nimber}(K_1)\} \\
&\qquad 1 \oplus 1 = 0 \\
&= 1
\end{aligned}
$$

In general: if we compute $\text{nimber}(K_0), \text{nimber}(K_1), \text{nimber}(K_2), \ldots$ in order, then we always use nimbers that we've already computed (because smaller)

– in Python, can do this with for loop:

```
k = {}
for n in range(0, 1000):
   k[n] = mex ([k[i] ^ k[n - i - 1] for i in range(n)] +
               [k[i] ^ k[n - i - 2] for i in range(n - 1)])
   print n, "-", k[ ]

def mex(nimbers):
   nimbers = set(nimbers)
   n = 0
   while n in nimbers:
      n = n + 1
   return n
```

960 – 4	972 – 4	984 – 4
961 – 1	973 – 1	985 – 1
962 – 2	974 – 2	986 – 2
963 – 8	975 – 8	987 – 8
964 – 1	976 – 1	988 – 1
965 – 4	977 – 4	989 – 4
966 – 7	978 – 7	990 – 7
967 – 2	979 – 2	991 – 2
968 – 1	980 – 1	992 – 1
969 – 8	981 – 8	993 – 8
970 – 2	982 – 2	994 – 2
971 – 7	983 – 7	995 – 7

periodic mod 12!
(starting at '72)
[Guy & Smith 1972]

DYNAMIC PROGRAMMING

How fast? to compute $\text{nimber}(K_n)$:

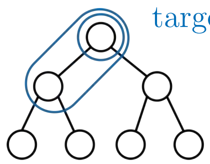
- look up $\approx 4n$ previous nimbers
- compute $\approx 2n$ nimsums (XOR)
- compute one mex on $\approx 2n$ nimbers
- call all this $\boxed{O(n)}$ work "order n"
- need to do this for $n = 0, 1, \ldots, m$

$$\Rightarrow \sum_{n=0}^{m} O(n) = O\left(\sum_{n=0}^{m} n\right) = O\left(\frac{m(m+1)}{2}\right) = O(n^2)$$

POLYNOMIAL TIME — GOOD

Variations: dynamic programming also works for:

- Kayles on a cycle
  (1 move reduces to regular Kayles ⇒ 2nd player win)
- Kayles on a tree:



- Kayles with various ball sizes: hit 1 or 2 or 3 pins
  (still 1st player win)

Cram: impartial Domineering

- board = $m \times n$ rectangle, possibly with holes
- move = place a domino (make 1×2 hole)
  Symmetry strategies: [Gardner 1986]
  - even × even: reflect in both axes
    ⇒ 1st player win
  - even × odd: play 2 center □s then reflect in both axes
    ⇒ 1st player win
  - odd × odd: OPEN who wins?

  Liner Cram = $1 \times n$ cram
  - easy with dynamic programming
  - also periodic [Guy & Smith 1956]

  - 1×3 blocks still easy with DP
  - OPEN : periodic?

  Horizontal Cram: 1 only
  ⇒ sum of linear crams!

  $2 \times n$ Cram: Nimbers OPEN Let's play!

  $3 \times n$ Cram: winner OPEN
  (dynamic programming doesn't work)

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