

# SP.268 – The Mathematics of Toys and Games

# Class Overview

## Instructors:

- Jing Li (class of '11, course 14 and 18C)
- Melissa Gymrek (class of '11, course 6 and 18)
- Supervisor: Professor Erik Demaine (Theory of Computation group at CSAIL)

## Requirements:

- Weekly attendance is mandatory!
- Occasional Readings
- Final projects

# Topics

- Theory of Impartial Games
- Surreal Numbers
- Linear Algebra and Monopoly
- Algorithms/Complexity of Games
- Dynamic Programming
- Artificial Intelligence Topics
- Rubik's Cube and Group Theory
- Probability Topics
- Games on Graphs
- NP-complete games
- Card Games
- Constraint Logic Theory
- Conway's Game of Life

# Theory of Impartial Games

Much of the game theory we will talk about will involve **combinatorial** games, which have the following properties:

- There are two players
- There is a finite set of positions available in the game
- Players alternate turns
- The game ends when a player can't make a move
- The game eventually *ends* (not infinite)

# Types of Games

Today's topic is **impartial games**:

- The set of allowable moves depends only on the positions of the game and not on which of the two players is moving
- Example impartial games: Jenga, Nim, sprouts, green hackenbush

In **partisan games**, the available moves depend on which player is moving. (GO, checkers, chess)

# The Game of Nim

- Much of the foundations for combinatorial game theory came from analyzing the game of Nim
- Here we use Nim to learn about types of game positions, nimbers, and to lead into an important combinatorial game theory theorem



# Nim: How to Play

- The game begins with 3 (or  $n$ ) piles, or *nim-heaps* of stones (or coins, or popsicle sticks...)
- Players 1 and 2 alternate taking off any number of stones from a pile until there are no stones left.
- In *normal play*, which is what we will look at, the player to take a stone wins.

# Example Game

Size of Heaps	Moves
A B C	
3 4 5	I take 2 from A
1 4 5	You take 3 from C
1 4 2	I take 1 from B
1 3 2	You take from B
1 2 2	I take entire A heap leaving two 2's
0 2 2	You take 1 from B
0 1 2	I take 1 from C leaving two 1's
0 1 1	You take 1 from B
0 0 1	I take entire C heap and win.



# Can we find a winning strategy?

- Yes! Nim has been solved for all starting positions and any number of heaps.
- (we will define what we mean by **solution** more rigorously later)
- We'll first do some work with game positions and "nimbers," and then apply them to finding a solution.

# Types of impartial game positions

There are two types of positions in an impartial game:

- **P Position:** secures a win for the **P**revious player (the one who just moved)
  - Example: a game of Nim with three heaps (1,1,0)
- **N Position:** secures a win for the **N**ext player
  - Example: a game of Nim with three heaps (0,0,1)
- **Terminal Position:** a position with no more available moves

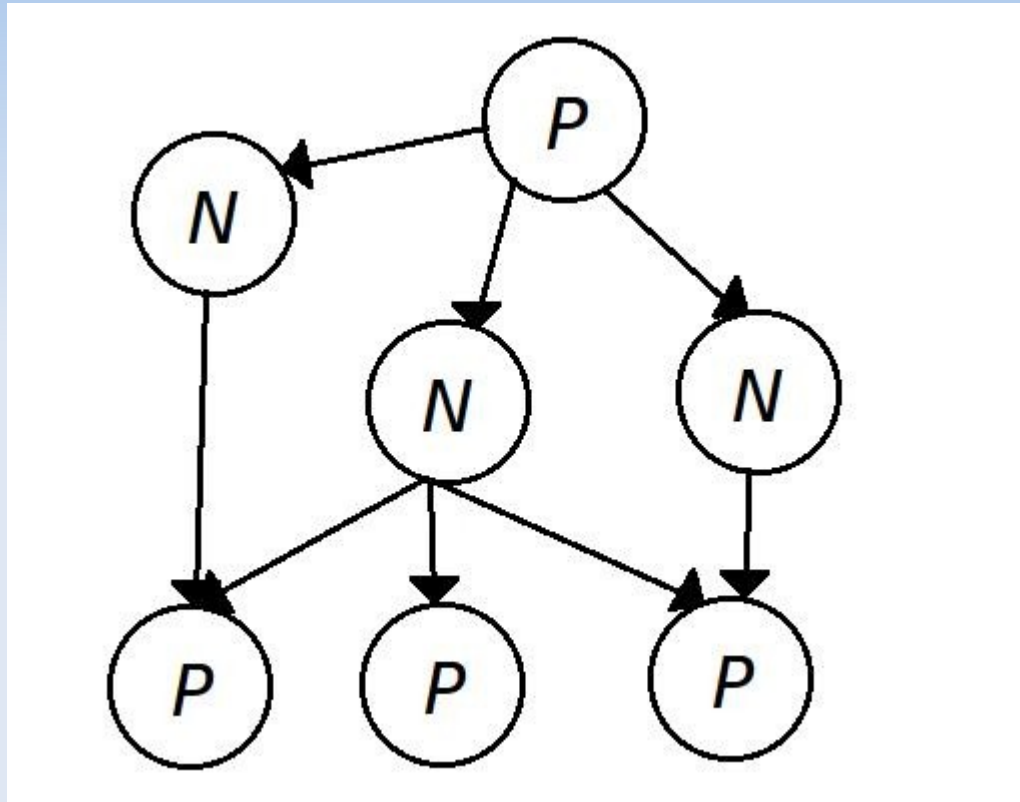
# Types of impartial game positions

- To determine whether a Nim (or any other impartial game) position is N or P, we work backwards from the end of the game to the beginning using **backwards induction**
  1. Label every **terminal position** as P
  2. Label every position that can reach a P position as N
  3. For positions that only move to N positions, label as P.
  4. At this point either all positions are labeled, or return to step 2 and repeat the process until all positions are labeled.

# Types of impartial game positions

- For example, in Nim, the only terminal position (in other games we might have many) is  $(0,0,0)$ .
- Any position  $(0,0,n)$  must be an N position, since the next player can just take the last heap in the next turn.

# Types of impartial game positions



# Practice with N and P positions

- Consider the Nim-like subtraction game in which you start with a pile of chips (note here we have only a single pile), and players alternate taking away either 1, 3 or 4 chips from the pile. The player to take the last chip loses.
- We see that 1, 3, and 4 must be N-positions, since the next player can take all the chips. 0 must be a P-position, since the player that last moved wins. 2 must be a P-position, since the only legal move to 2 is from an N-position (2 can move only to 1). Then 5 and 6 must be N positions since they can be moved to 2... continue the analysis up to  $x = 14$ :

# Practice with N and P positions

Finish the game configuration assignments...

X	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
---	---	---	---	---	---	---	---	---	---	---	----	----	----	----	----

Pos	P	N	P	N	N	N	N								
-----	---	---	---	---	---	---	---	--	--	--	--	--	--	--	--

# Practice with N and P positions

X	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Pos	P	N	P	N	N	N	N	P	N	P	N	N	N	N	P

Note: this sequence of Ns and Ps (PNPNNNN) continues forever. In fact, almost all subtraction games have similar periodic sequences of N and P values.



# Number Arithmetic

- The key operation we will use in our solution to Nim will be **number addition**, which is just binary addition without carrying. We can also describe this as writing out the numbers to add in binary, then XORing the numbers bit by bit.
- We denote the nim-sum operation by the following symbol (but we'll usually just use a "+" sign here):  
$$8 \oplus 8 = 0$$

# Binary logic refresher

- Recall, to read/write numbers in binary:

The bits in a binary number represent powers of 2:

$$0 \quad 1 \quad 0 \quad 1 \quad 1 = 2^3 + 2^1 + 2^0 = 11$$

$$2^4 \quad 2^3 \quad 2^2 \quad 2^1 \quad 2^0$$

- XORing:

$$1+1 = 0, \quad 1+0 = 1, \quad 0+0 = 0$$

# Number Addition Examples

$$\begin{array}{r} 3 \\ + 5 \\ \hline 7 \end{array} \qquad \begin{array}{r} 011 \\ 100 \\ \hline 111 \end{array}$$

$$\begin{array}{r} 4 \\ + 4 \\ \hline 0 \end{array} \qquad \begin{array}{r} 010 \\ 010 \\ \hline 000 \end{array}$$

# The Nim addition table

+	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	0	3	2	5	4	7	6	9	8	11	10	13	12	15	14
2	2	3	0	1	6	7	4	5	10	11	8	9	14	15	12	13
3	3	2	1	0	7	6	5	4	11	10	9	8	15	14	13	12
4	4	5	6	7	0	1	2	3	12	13	14	15	8	9	10	11
5	5	4	7	6	1	0	3	2	13	12	15	14	9	8	11	10
6	6	7	4	5	2	3	0	1	14	15	12	13	10	11	8	9
7	7	6	5	4	3	2	1	0	15	14	13	12	11	10	9	8
8	8	9	10	11	12	13	14	15	0	1	2	3	4	5	6	7
9	9	8	11	10	13	12	15	14	1	0	3	2	5	4	7	6
10	10	11	8	9	14	15	12	13	2	3	0	1	6	7	4	5
11	11	10	9	8	15	14	13	12	3	2	1	0	7	6	5	4
12	12	13	14	15	8	9	10	11	4	5	6	7	0	1	2	3
13	13	12	15	14	9	8	11	10	5	4	7	6	1	0	3	2
14	14	15	12	13	10	11	8	9	6	7	4	5	2	3	0	1
15	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0

# Solution to Nim

- Now we can finally find the solution to Nim!
- Notice that if we take the nim-sum of all the nim-heaps, at the end of the game the nim-sum will always be 0 (since we are adding  $0+0+0$ ).
- But there are other points in the game when the nim-sum will be 0, for instance  $(0,n,n)$ , since any  $n + n = 0$ .

# Solution to Nim

- Theorem: The winning strategy in normal play Nim is to finish every move with a nim-sum of 0.

To prove this we will first prove two lemmas...

# Solution to Nim

- Lemma 1: If the nim-sum is 0 after a player's turn, then the next move must change the nim-sum to be nonzero.
- Proof... (in lecture notes and given in class)

# Solution to Nim

- Lemma 2: It is always possible to make the nim-sum 0 on your turn if it wasn't already 0 at the beginning of your turn.
- Proof... (in lecture notes and given in class)



# Solution to Nim

- Proof of the original theorem:  
If you start off by making your first move so that the nim-sum is 0, then on each turn, by lemma 1, your opponent has no choice but to disturb the sum and make it non-zero. By lemma 2, you can always move to set it back to 0. Eventually on your turn there will be no stones left and the game will have a nim-sum of 0, and you will win! (If the nim-sum starts at 0 and you go first, you must hope your opponent messes up at some point, or else he has the winning strategy...)

# Solution to Nim

- From our analysis above, we can see that any nim position in which the nim-sum of the heaps is 0 is a P-position, otherwise it is an N-position.

# Nim – Practical Strategy

- Whenever possible, reduce the heaps to two non-zero heaps containing the same number of coins each. This obviously has a nim-sum of 0. Now just mimic your opponent's move each time on the opposite heap to keep the two heaps equal until you are able to take the final coin.
- Since doing binary addition is kind of hard in your head for large numbers, an easier strategy is to try to always leave even subpiles of the subpowers of 2, starting with the largest power possible (essentially the same as binary addition, but easier to think about)

# Variations on (disguises of) Nim

Why are these the same as Nim?

- Poker Nim: Like regular Nim, but now a player can either add more chips to a heap or subtract chips from heap (known as **bogus nim heaps**). What is the winning strategy?
- Northcott's Game: checkerboard with one white and one black checker on each row. Players alternate moving their color checker piece any number of positions along a row but may not jump over or onto another checker. Players move until no one can make another move. (this game is neither impartial nor finite but we can use our nim strategy)

# Sprague-Grundy Theorem

- Now we'll use Nim to help us derive the fundamental theorem of impartial games, the Sprague-Grundy Theorem.
- We'll start by using graphs to describe impartial game positions.

# From Games to Graphs

- Let a game consist of a graph  $G = (X, F)$  where:
  - $X$  is the set of all possible game positions
  - $F$  is a functions that gives for each  $x$  in  $X$  a subset of possible  $x$ 's to move to, called followers. If  $F(x)$  is empty, the position  $x$  is terminal.
  - Start position is  $x_0$  in  $X$ . So player 1 moves first from  $x_0$ .
  - Players alternate moves. A position  $x$ , the player chooses from  $y$  in  $F(x)$ .
  - The player confronted with the empty set  $F(x)$  loses.
- We call our graph **progressively bounded** if from every start position  $x_0$ , every path has finite length. So the graph is finite and acyclic.

# From Games to Graphs

- The following is an example of the graph for the Subtraction Game that we saw before (starting with 10 coins in the heap. The person that makes the total number of coins 0 on his or her turn wins.

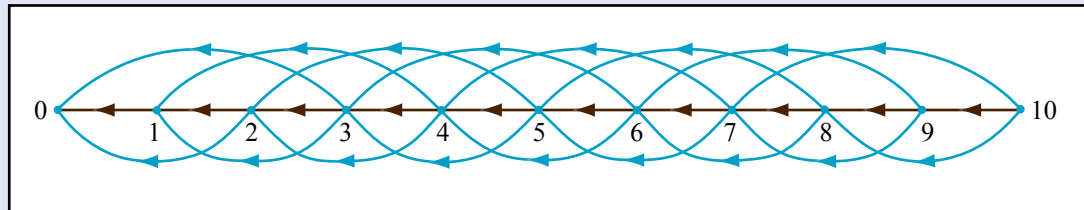


Image by MIT OpenCourseWare.

# The Sprague Grundy Function

- The **Sprague-Grundy** function of a graph  $G = (X, F)$  is a function  $g$  defined on  $X$  that takes only non-negative integer values and is computed as follows:

$$g(x) = \min\{n \geq 0 : n \neq g(y) \text{ for } y \in F(x)\}$$



# The Sprague-Grundy Function

- In words, the Sprague-Grundy function (I'll call it SG) is the smallest non-negative value not found among the SG values of the followers of  $x$ .
- This is formally known as the **mex** function, or **Minimum Excluded Value**.
- Examples
  - $\text{mex}(\{2,4,5,6\}) = 0$
  - $\text{mex}(\{0,1,2,6\}) = 3$

# The Sprague-Grundy Function

- So we can rewrite the SG function as:

$$g(x) = \text{mex}\{g(y) : y \in F(x)\}$$

- Notice this function is defined recursively. That is, the definition of  $g(x)$  uses  $g$  itself. So we'll need some base cases. Set all terminal nodes  $x$  to have  $g(x) = 0$ . Then any nodes that have only terminal nodes as followers have  $g(x) = 1$ . In this way we can work our way through the graph until all nodes are assigned an SG value.

# SG Function Example

- Try assigning SG values to the graph below. Start by assigning each node N or P:

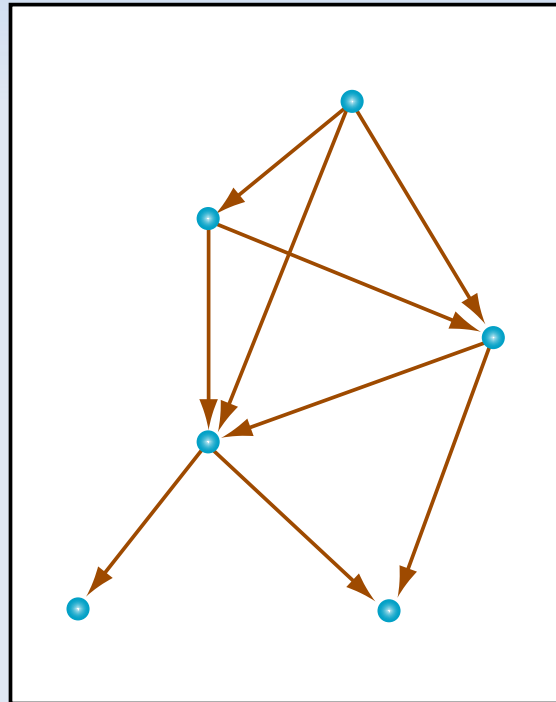


Image by MIT OpenCourseWare.

# SG Function Example

- You should end up with the following:

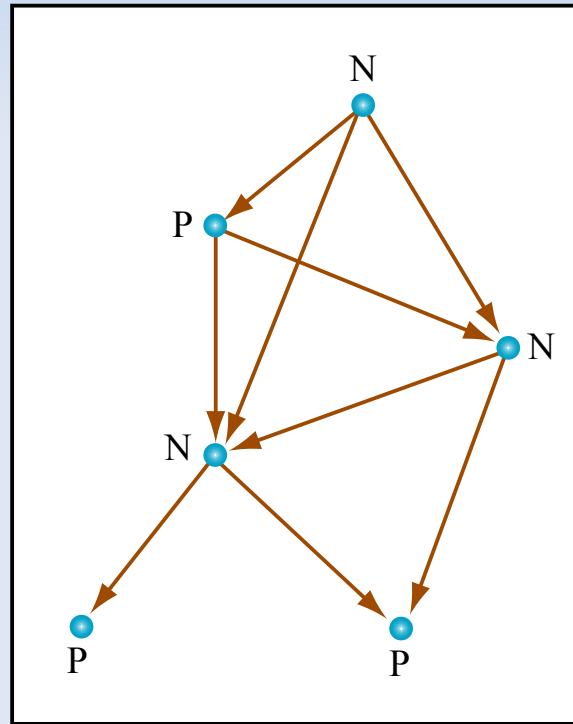


Image by MIT OpenCourseWare.

# SG Function Example

- Now we can label all the SG values. Start with the terminal positions set to 0. Any node pointing to a terminal gets a 1. The node pointing to only 0 and 1 should be labeled 2, and so forth.

# SG Function Example

- You should end up with the following SG assignments:

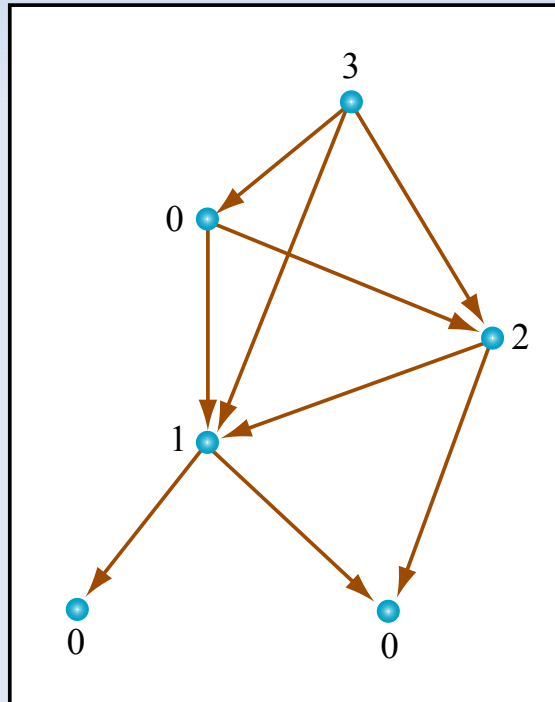


Image by MIT OpenCourseWare.

# SG Function

- Notice that all vertices that have an SG value of 0 are P positions! All others are N.
- It seems that a good strategy on such a graph-game would be to move to a vertex with  $g(x) = 0$ .

# SG Example 2

- Consider the "21 subtraction game." You and a friend start with 21 coins, and then take turns taking up to 3 coins away at a time. The person to take the last coin wins.
- We can see that the position with 4 coins left is a P position. The next player must reduce the pile to some number within the range 1-3, and so the player after him can win. Below are the SG values for this game:

x	0	1	2	3	4	5	6	...
g(x)	0	1	2	3	0	1	2	...

or in other words,  $g(x) = x \bmod 4$ .

- For the game of Nim with one heap, the function is just  $g(x) = x$ .



# Adding Games (Graphs)

- An advantage of using the SG function is that we can break up games into smaller parts and *add* their graphs together to make one big game.
- We call this the **disjunctive sum** of two games.
- To take the disjunctive sum of games  $G$  and  $H$ , players can move on their turn in either the game  $G$  or the game  $H$ , and the entire game is over when both  $G$  and  $H$  are at terminal positions.

# Adding Games

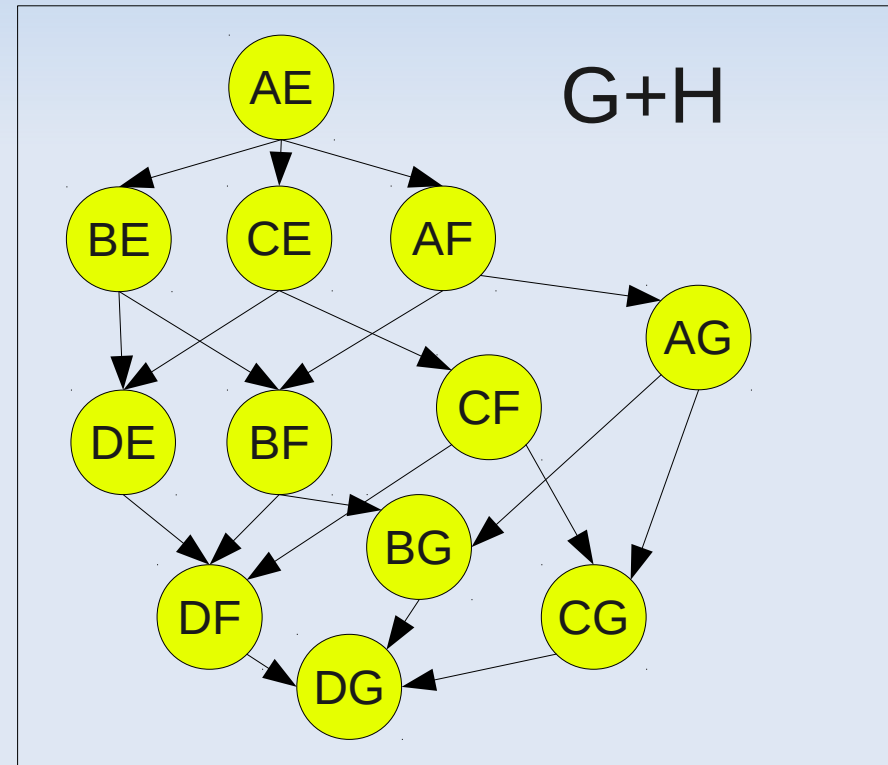
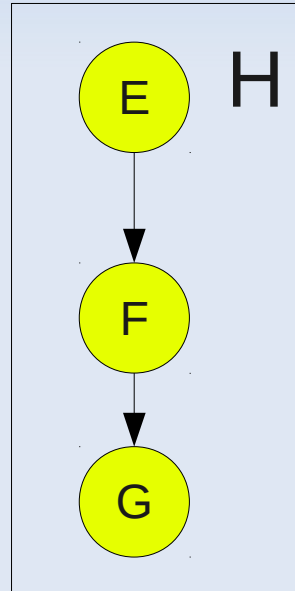
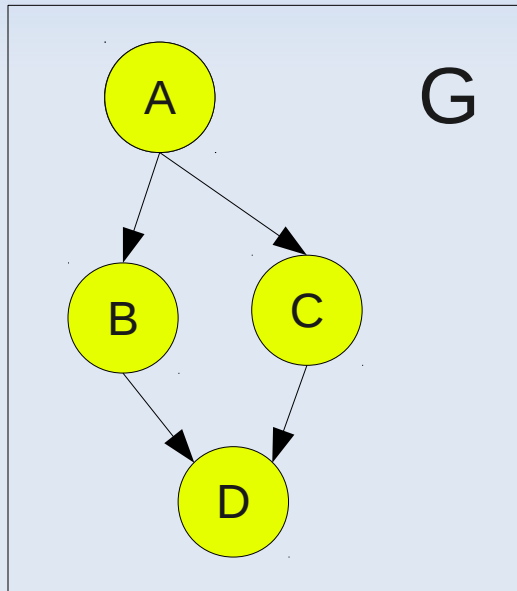
- To sum the games  $G_1 = (X_1, F_1)$ ,  $G_2 = (X_2, F_2)$ , ...,  $G_N = (X_N, F_N)$ ,

$G(X, f) = G_1 + G_2 + \dots + G_N$ , where

- $X = X_1 \times X_2 \times \dots \times X_N$  (or the set of all n-tuples such that  $x_i$  in  $X$  for all  $i$ ).
- The maximum number of moves is the sum of the maximum number of moves in each component game.

# Adding Games

- Here is an example of adding two games on graphs, G and H:



# SG Theorem

- If  $g_i$  is the Sprague-Grundy function of  $G_i$ ,  $i = 1, 2, \dots, n$ , then  $G = G_1 + \dots + G_n$  has Sprague-Grundy function  $g(x_1, \dots, x_n) = g_1(x_1) + g_2(x_2) + \dots + g_n(x_n)$
- Proof... (outlined here and shown in lecture notes and in class)

# SG Theorem

- Let  $x = (x_1, \dots, x_n)$  be an arbitrary point of  $X$ . Let  $b = g_1(x_1) + \dots + g_n(x_n)$ . We are to show two things for the function  $g(x_1, \dots, x_n)$ :
  - For every non-negative integer  $a < b$ , there is a follower of  $(x_1, \dots, x_n)$  that has a  $g$ -value  $a$ .
  - No follower of  $(x_1, \dots, x_n)$  has  $g$ -value  $b$ .
- Then the SG value of  $x$ , being the smallest SG-value not assumed by one of its followers, must be  $b$ .

# Games to try

- Variations on Nim
- Green Hackenbush
- Sprouts
- more!

# Further Reading

- Winning Ways

MIT OpenCourseWare  
<http://ocw.mit.edu>

SP.268 / ESG.SP268 The Mathematics in Toys and Games  
Spring 2010

For information about citing these materials or our Terms of Use, visit: <http://ocw.mit.edu/terms>.