Graphical Integration Exercises
Part Four:
Reverse Graphical Integration

Prepared for the
MIT System Dynamics in Education Project
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1. ABSTRACT

Reverse graphical integration is a method which allows one to quickly estimate the values of a stock's net flow based on a graph of the stock's behavior over time. This technique is useful in cases where a stock's behavior is known and one would like to find the associated flow behavior. This paper covers two simple ways to determine information about a net flow which can be used to draw its graph. The reader should be familiar with the concepts covered in *Graphical Integration Exercises Part Three: Combined Flows*.[1]

[1]Kevin Agatstein and Lucia Breierova, 1996. *Graphical Integration Exercises Part Three: Combined Flows* (D-4596), System Dynamics in Education Project, System Dynamics Group, Sloan School of Management, Massachusetts Institute of Technology, March 14, 32 pp.

2. INTRODUCTION

When one deals with real systems, the data which describes model variables comes in the form of stocks. The height of a high tide, the number of customers, the size of a city, the balance of a bank account, and the national debt are all stocks. Even values which are thought of as rates come in the form of stock measurements over time. The rate of snowfall is determined by checking the amount of snow which has fallen over time. A typist's speed can be determined by looking at the number of words which have been typed over time. Rates are not instantaneously measurable; all information about rates comes to us through stock values.

Determining a rate from real world stock values is helpful when writing rate equations for a model. Being able to sketch a graph of a flow from a graph of a stock can make working with stock data more informative.

The first three papers in this series discussed how to look at inflows and outflows to determine the behavior of a stock. This paper covers methods for doing the reverse: graphically determining flow information from a stock graph.

3. REVERSE GRAPHICAL INTEGRATION

3.1 Overview

The method of determining rate information from stocks, which this paper shall refer to as *reverse graphical integration*, is also known in calculus as the graphical form of *differentiation*. Although reverse graphical integration may be done with mathematical equations or with calculations on a computer, for the purposes of understanding a system's flows, this level of accuracy is time consuming and unnecessary. A graph pad and a few guidelines are all that one needs to effectively determine a stock's net flow.

The amount of information about a stock's inflows and outflows which can be determined from the stock is limited. When several flows are combined together to change the value of a stock, information about the characteristics of the individual flows is lost. For this reason, reverse graphical integration will give the behavior of the stock's net flow,[2] but it will not identify the specific inflows and outflows which may be associated with the system. For example, one could determine that the rate of the world

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[2] For more information on net flows, see *Graphical Integration Exercises Part Three: Combined Flows* (D-4596).

population's growth is increasing from the world population stock's behavior but one could not determine whether this is the result of a higher birth rate in China or in Mexico.

Reverse graphical integration takes the graph of a stock's behavior over time and produces a graph of the net flow associated with the stock. An easy way to think of this is to divide the stock behavior graph into several vertical sections. Within each of these sections one can calculate the slope of the stock. These slope values provide a rough sketch of the net flow of the stock. This method is the basis of all methods of reverse graphical integration. For more accuracy, all that one must do is divide the graph into smaller vertical sections.[3] However, this is cumbersome. In practice it is easier to divide the graph into sections to get the general shape. Then, if it is necessary, one can calculate the exact net flow at certain points to add accuracy to the net flow graph. If the points picked cover the behavior well, then connecting the dots to produce the graph is a simple operation. Though this may sound confusing, it is not. The following example demonstrates how simple slope calculations may be used to estimate the net flow.

3.2 Example 1: Using Intervals with Linear Stock Behaviors

The Compton widget factory makes widgets and ships them to distributors around the nation. Figure 1 shows a graph of their widget inventory over the past three quarters:

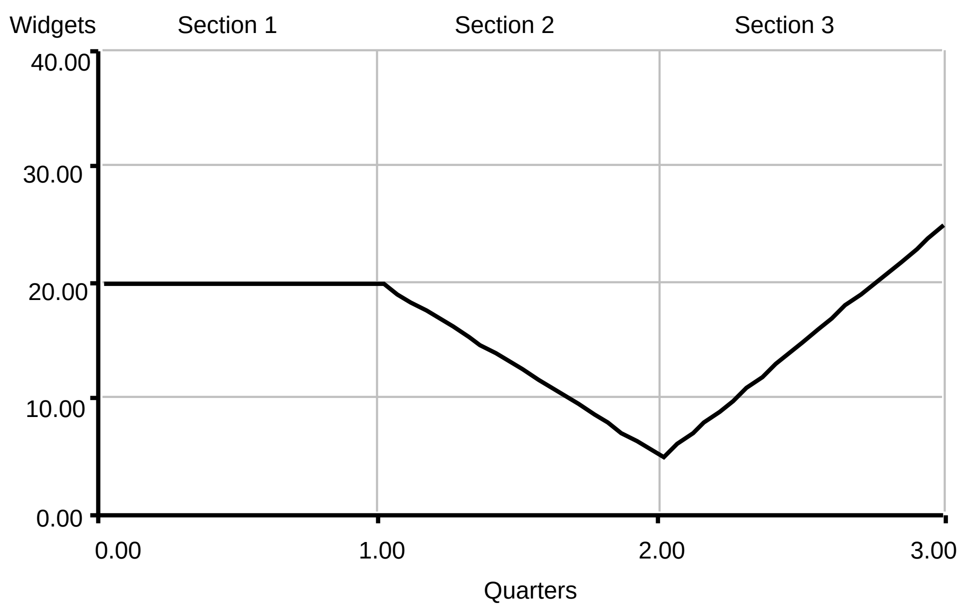

Figure 1: Widget inventory behavior over three months

The widget inventory graph above is the graph of a stock. To aid in determining the graph of the net flow into the stock, the stock graph has been divided into three

[3] Calculus achieves perfect accuracy by dividing the graph into infinitely many sections.

equally spaced sections. In Section 1, the value of the stock remains constant. This indicates that the difference between the initial and final stock values within Section 1 is zero. Hence we say that the value of the net flow over the first quarter is zero even though the actual inflows and outflows may have been in the hundreds of widgets per quarter. All that the graph shows is that the sum of the inflows and sum of the outflows of widgets were equal.

Section 2 is slightly different from Section 1. The stock of widgets decreases linearly from 20 widgets to 5 widgets. This is a difference of negative 15 widgets (final widgets minus initial widgets), thus the net flow over Section 2 is negative 15 widgets per quarter. In Section 3 the inventory changes again, from 5 widgets to 25 widgets. This is a change of 20 widgets per quarter. The results of these measurements can then be plotted on a graph.

The change in stock value over time is also known as the slope.[4] Slope is a quantity which is measured by the change in the value of the vertical scale divided by the change in the value of the horizontal scale for any two given points on a graph. The net flow graph shows the slope of the stock over time, so by definition the change in the stock per unit time is the “net flow” as well as the slope.

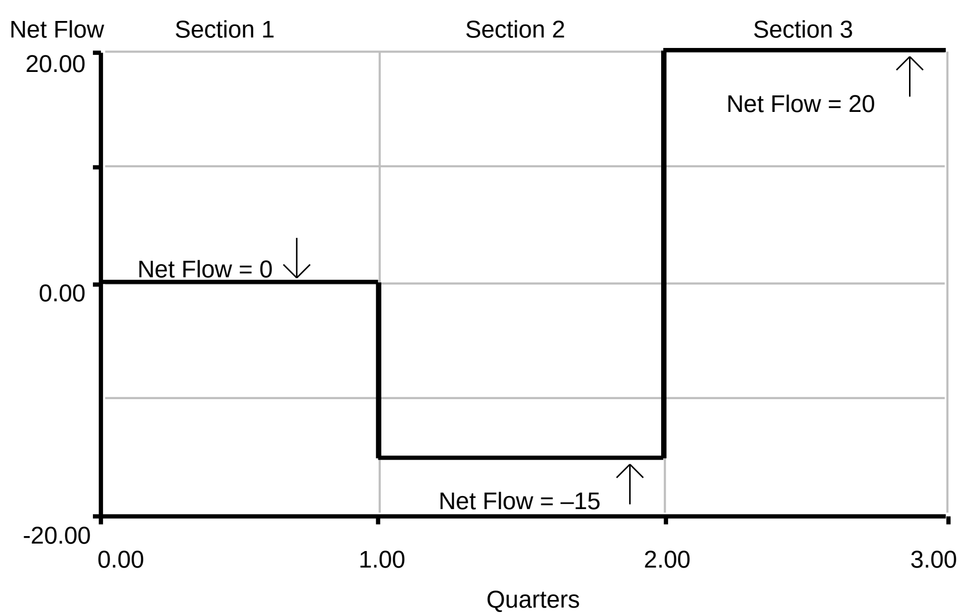

Figure 2: Net Flow of Widgets

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[4] For more information on slopes, see: Kevin Agatstein and Lucia Breierova, 1996. *Graphical Integration Exercises Part Two: Ramp Functions* (D-4571), Systems Dynamics in Education Project, System Dynamics Group, Sloan School of Management, Massachusetts Institute of Technology, March 25, 25 pp.

The results of the reverse graphical integration of the widget inventory are shown in Figure 2. Several inferences, delineated in Table 1, can be made from the results. Remember:

Net Flow = Inflow - Outflow.

When the value of a stock is increasing, the inflow must be greater than the outflow, so the net flow will be positive. Likewise, when the value of the stock is decreasing, the outflow will be greater than the inflow so the net flow of the stock will be negative.

Table 1: Summary of net flow behavior for a linearly changing stock

Stock Behavior	Net Flow Behavior
Constant	Over a period where the value of a stock does not change, such as Section 1, the net flow will have a value of zero.
Increasing Linearly	Over a section where the value of the stock increases, as in Section 3, the net flow will be constant and positive.
Decreasing Linearly	Over a section where the value of the stock decreases, as in Section 2, the net flow will be constant and negative.

3.3 Example 2: Using Intervals for More Complicated Behaviors

Shreve's Gator Farm buys and breeds alligators. Inventory records produced the graph in Figure 3.

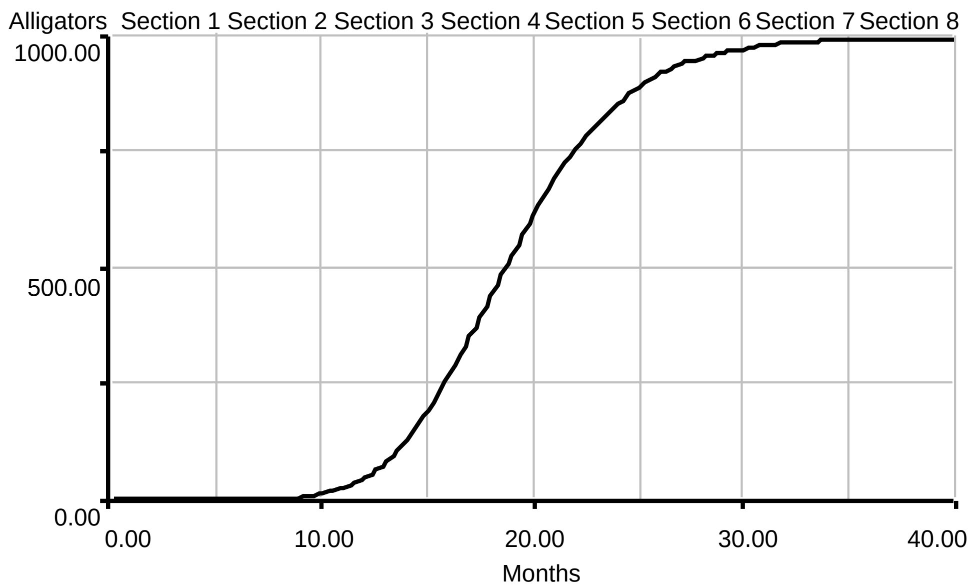

Figure 3: Alligator Inventory

The above graph of Alligator inventory shows S-shaped growth. It has been divided into smaller sections than before because of the greater complexity of the behavior. When faced with a complicated graph, a good technique is to simply break it up into small pieces.

To graph this net flow, assume that the stock makes approximately a straight line within each section. Graphing the flow then becomes a matter of calculating the slopes and plotting them. The calculations proceed as follows:

By definition,

$$Slope = \frac{F - I}{T}$$

where I is the *initial* value of the stock at the beginning of the section, F is the *final* value of the stock at the end of the section, and T is the length of the section. In this example, T is 5 months for each section.

In Section 1, the value of the stock for the whole section is 10 alligators,[5] so the slope is:

$$\frac{F - I}{T} = \frac{10 - 10}{5} = 0 \text{ alligators/month.}$$

In Section 2, the value of the stock at the beginning of the section is 10 alligators, and the value at the end of the section is 24 alligators. The slope is:

$$\frac{F - I}{T} = \frac{24 - 10}{5} = 2.8 \text{ alligators/month.}$$

The rest of the calculations proceed in the same manner and are summarized in Table 2.

[5] The initial value of the stock is 10 alligators.

Table 2: Summary of the slope calculations for alligator inventory

Section	Change in Stock (alligators)[6]	Change in Time (months)	Slope (alligators/month)
Section 1	0	5	0
Section 2	14	5	2.8
Section 3	180	5	36
Section 4	418	5	83.6
Section 5	278	5	55.6
Section 6	83	5	16.6
Section 7	16	5	3.2
Section 8	1	5	.2

The slope information can be plotted on a graph to reveal the flow behavior as in Figure 4.

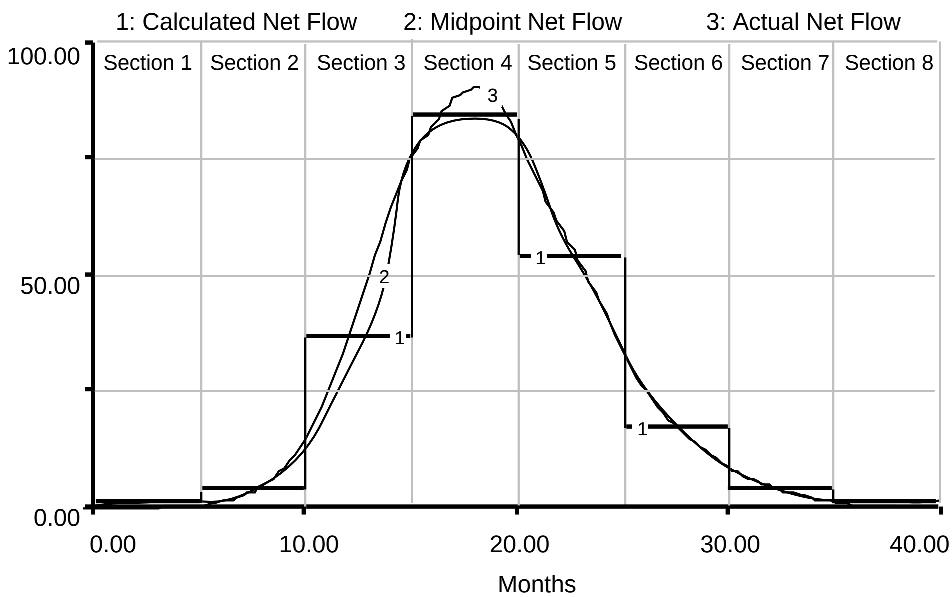

Figure 4: Net flow behavior of Alligator Inventory

Figure 4 shows the net flow behavior from the calculations in the table along with the actual net flow behavior. Drawing a curve which connects the midpoints of the line segments usually makes a good approximation of the actual net flow. The section calculations are helpful, but do not give actual points on the graph. Tangent lines, covered in the next section, are needed to find these specific points.

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[6] The numbers in this table are more accurate than one would be able to pull directly off of the Stella graph because they were taken from a Stella table.

It should be noted that sections do not need to be evenly spaced. As long as T is adjusted for the width of each section, the flow will be scaled correctly. Varying the spacing can save calculations in some cases. If the stock is fairly linear over a certain part of the graph, then using one section will be as effective as using multiple sections over that part. Over parts of the graph that curve, more sections should be used because the slope is changing quickly. The greater number of sections will add more definition to the net flow in this area.

3.4 Example 3: Using Tangent Lines

When it is necessary to know the maximum value of the flow to a greater accuracy, tangent lines[7] may be used. Take the graph of the length of a work week in a company with a variable work week policy in Figure 5.

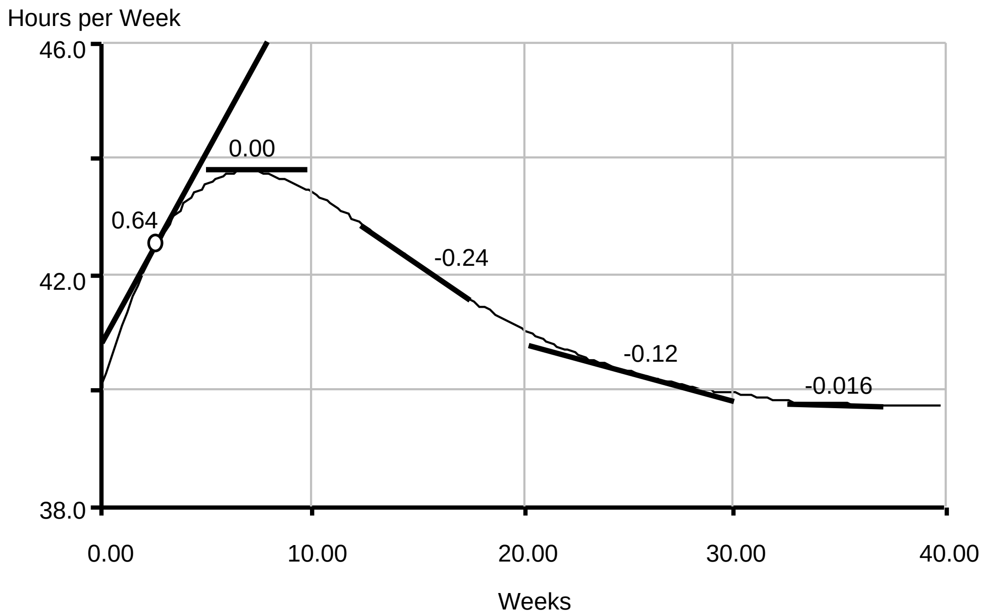

Figure 5: Length of Work Week

The graph shows both the length of the average work week and a series of lines drawn at various points on the curve. These are called tangent lines because they are parallel to the stock line at the point where the two touch. Essentially these lines have the same slope as the stock line at that point. This slope is also the value of the net flow at that point in time.

There is no real trick to drawing tangent lines. One should simply draw a straight line which is parallel to the graph at the point where the graph and the line connect. As

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[7] For more information on tangent lines, see *Graphical Integration Exercises Part Two: Ramp Functions* (D-4571).

long as the tangent touches the graph at the point where one is calculating the slope and the tangent line does not cross over the graph line, the tangent line is probably a good one. In some cases the tangent line may intersect the stock at a later or earlier point. Such secondary intersections should simply be ignored. Finding the slope of a tangent line is the same calculation covered in Example 2. Since the slope along the entire tangent line is constant, simply pick two points on the tangent line to use for reference. In Figure 5, the first tangent line has been extended to the sides of the graph for convenience. In the case of this line, it begins with a height of 40.9 hours/week at the 0 week mark. It leaves the edge of the graph at a height of 46 hours/week and a distance of 8 weeks. Recall that

$$Slope = \frac{F - I}{T}.$$

In this case, T is 8 weeks because it is the horizontal distance covered between the two reference points. As before, F is the final height, or 46 hours/week, and I is the first height, or 40.9 hours/week. The resulting slope is

$$\frac{F - I}{T} = \frac{46 - 40.9}{8} = \frac{5.1}{8} = 0.6375$$

which we can round to 0.64. The other tangent lines are calculated in the same manner. Table 4 gives times and net flows calculated using these tangent lines.

Table 3: Summary of tangent line slopes

Time	Net Flow[8]
2 Weeks	0.64 Hours/(Week*Week)
7 Weeks	0.00 Hours/(Week*Week)
14 Weeks	-0.24 Hours/(Week*Week)
25 Weeks	-0.12 Hours/(Week*Week)
35 Weeks	-0.016 Hours/(Week*Week)

These points can then be used in conjunction with the previous sectioning technique to produce a more accurate graph of the net flow. Instead of drawing a series of steps, the points given by the tangent lines can be used as references so that the graph may be rounded to a more realistic shape. The results of this process and the previously described section method are shown in Figure 6.

[8] Since the length of the work week is measured in hours per week, the change in this value is measured in Hours per Week per Week, or, as shown above, Hours/(Week*Week).

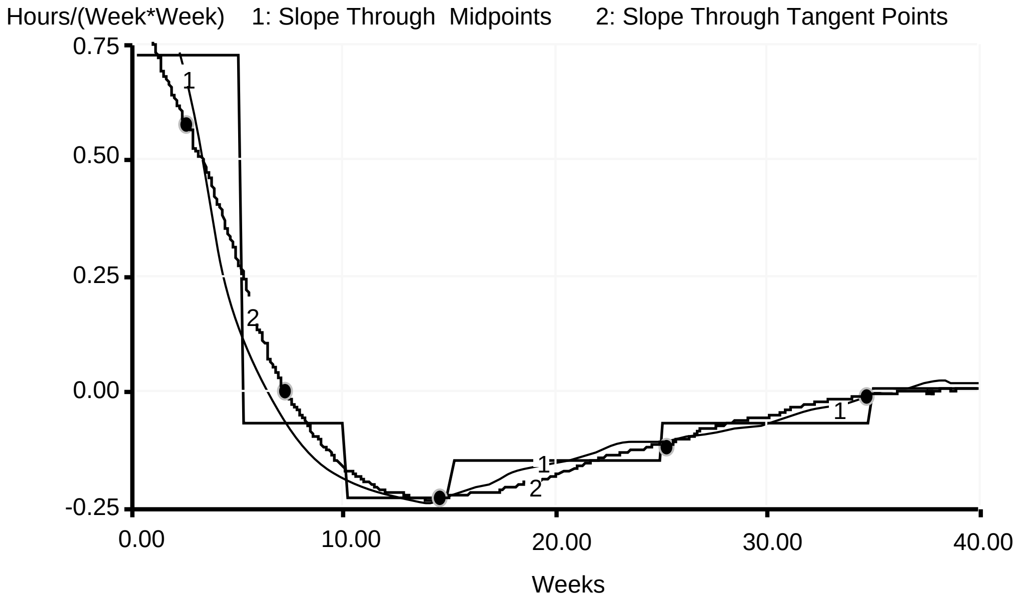


Figure 6: Net flow using tangent lines and sectioning

It can be seen that the section based method for determining the slope, while not as exact as the tangent method, can still work well. Using tangent lines to determine the slope at certain points, represented by the black circles, adds the extra points needed to adjust the curve more accurately. An intuition for the behaviors that a graph may exhibit is also helpful in producing a good graph of the net flow.

4. CONCEPT REVIEW

1. Reverse graphical integration takes a stock behavior and gives the net flow associated with that behavior.
2. Reverse graphical integration can determine the sum of the stock's inflows and outflows but it cannot determine the actual inflows and outflows.
3. When a stock is increasing, its net flow is positive; when a stock is decreasing its net flow is negative.
4. Split the graph into sections and calculate the slope within each section to get an idea of the shape and dimension of the net slope.
5. Draw tangent lines to determine specific values of a net flow and add accuracy to a sketch.

5. CONCLUSION

The techniques demonstrated here do a good job of sketching the net flow of any stock. When modeling, keep these in mind and make use of them when looking at stock behaviors. Knowing what the net flow of a stock looks like can save time and trouble when de-bugging a model in progress or understanding a new one. With practice, one can develop an intuition for the general shapes of stocks and their net flows.

6. EXERCISES

6.1 Exercise 1: Joe's Bank Account

Joe has been keeping an eye on his bank account. He is interested in finding out the reasons for the changes in its behavior. His balance for the last 40 days is shown in Figure 7.

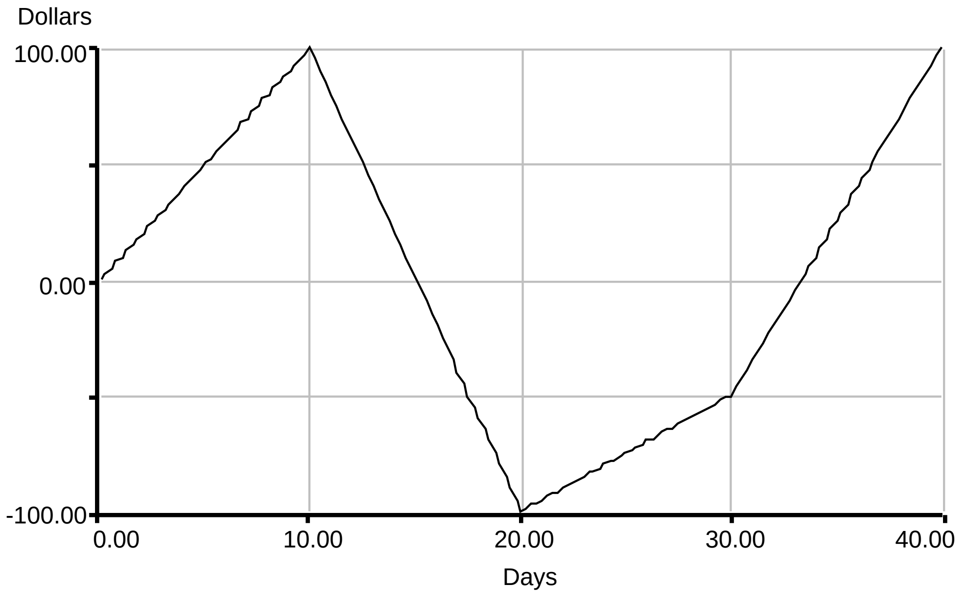
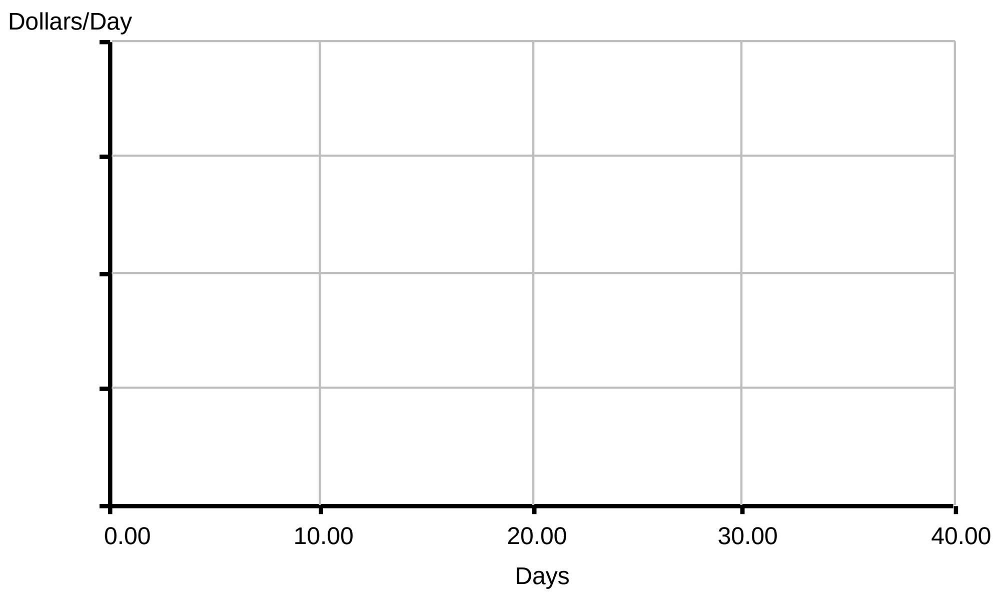


Figure 7: Joe's bank account balance

Use reverse graphical integration to draw the net flow associated with his bank balance on the graph pad provided. Remember that answers may always be checked by graphically integrating the net flow to confirm that the result is the stock behavior.



6.2 Exercise 2: Golf Course Development

Figure 11 is a graph of the development of the golf course from Beginner Modeling Exercises 5.[9]

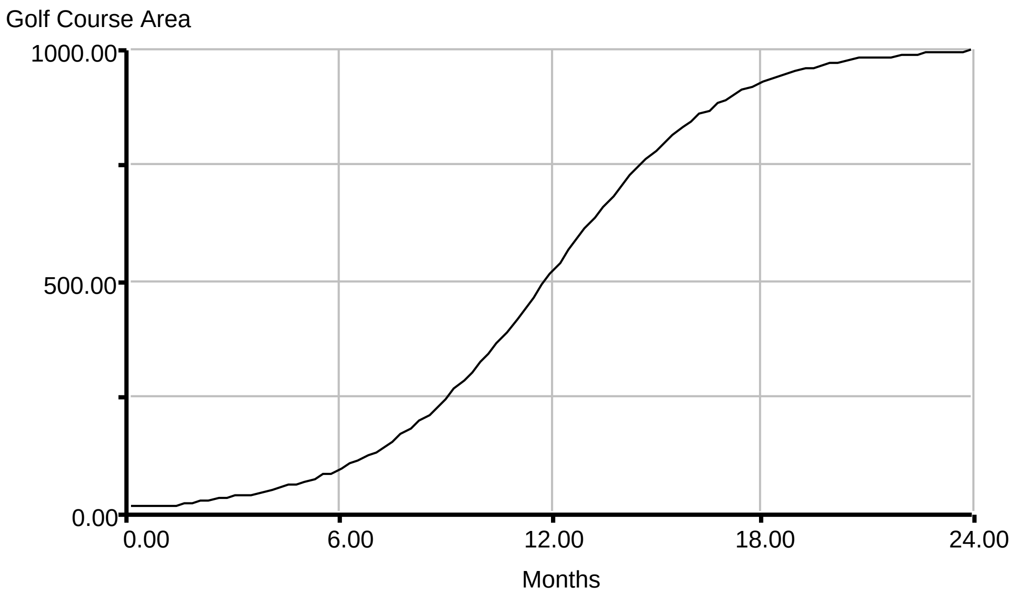
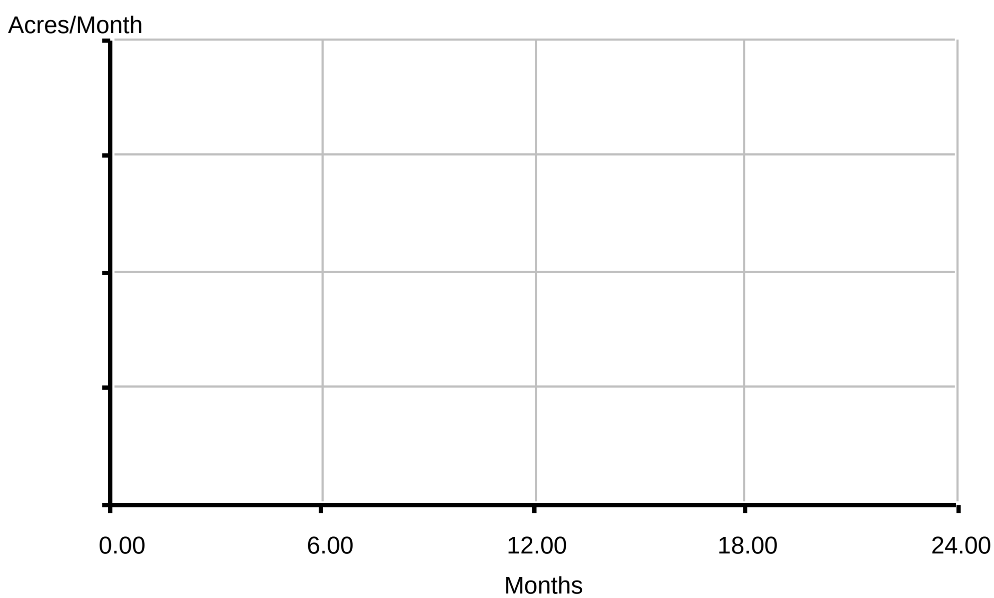


Figure 8: Golf course development

Draw a graph of the rate at which the course was developed on the graph pad provided.



[9]Laughton Stanley and Helen Zhu, 1996. *Beginner Modeling Exercises 5: Mental Simulation of Combining Feedback in First Order Systems* (D-4593), System Dynamics in Education Project, System Dynamics Group, Sloan School of Management, Massachusetts Institute of Technology, May 28, 1996.

6.3 Exercise 3: Overshoot and Collapse

One of the generic behaviors of systems is overshoot and collapse, as seen in the fish banks model, or with the introduction of a popular product such as an Atari system. Figure 9 is a graph of an overshoot and collapse model of a population of deer caused by the removal of natural predators.

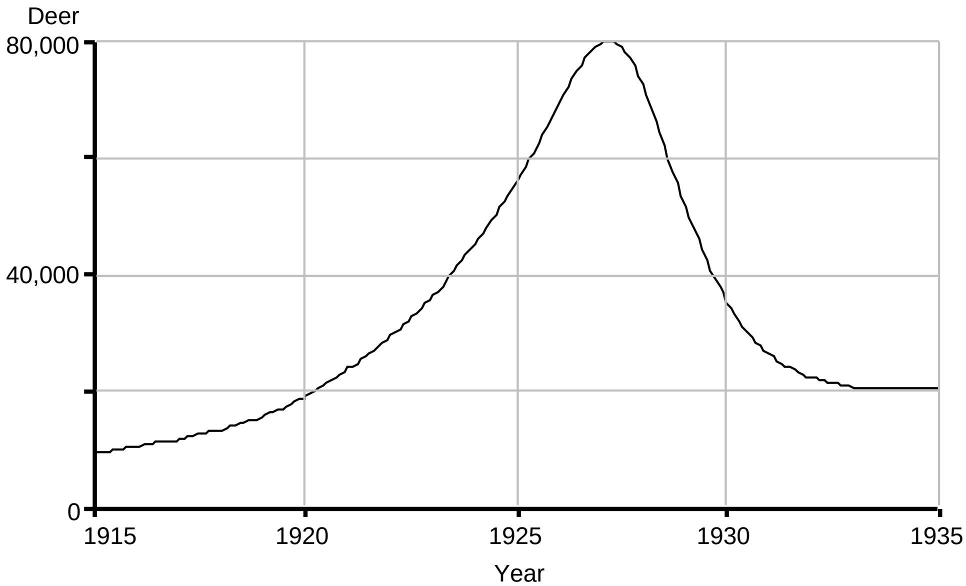
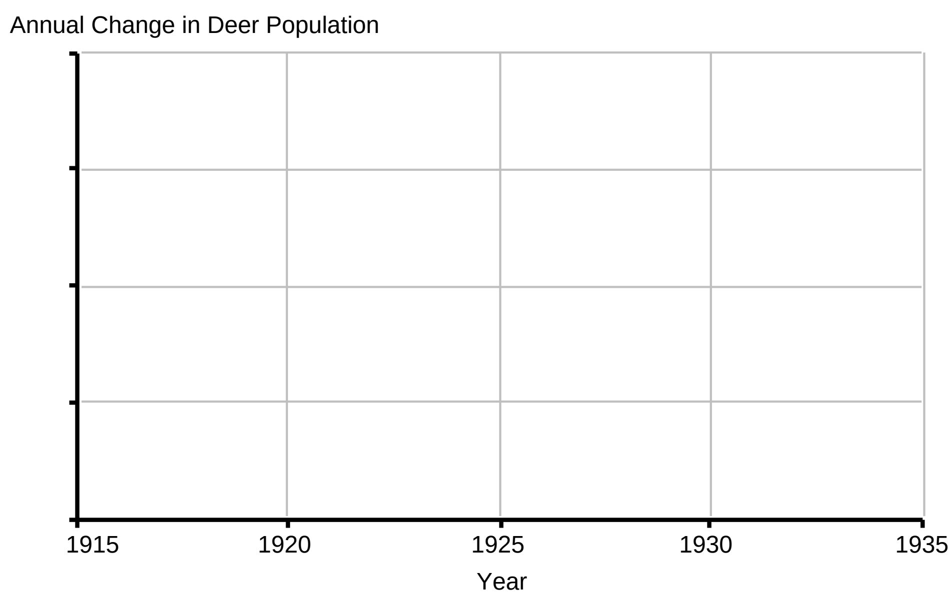


Figure 9: Overshoot and collapse in a deer population model

Draw a graph of the rate at which the deer population changed on the blank graph pad provided.



7. EXERCISE SOLUTIONS

7.1 Solutions to Exercise 1: Joe's Bank Account

The stock behavior of Joe's balance is made up of linear increases and decreases. Because slope is constant along a straight line, it is easiest to divide up the graph into sections at the points where the lines change. This occurs every 10 days. Within each section, all that is needed is a simple slope calculation. For example, the slope of the third section, between 20 and 30 days, would be calculated as follows:

$$Slope = \frac{F - I}{T} = \frac{(-50) - (-100)}{10} = \frac{50}{10} = 5 \text{ Dollars/Day.}$$

The slope between 30 and 40 days is calculated in the same manner:

$$Slope = \frac{F - I}{T} = \frac{100 - (-50)}{10} = \frac{150}{10} = 15 \text{ Dollars/Day.}$$

The resulting graph is shown in Figure 10.

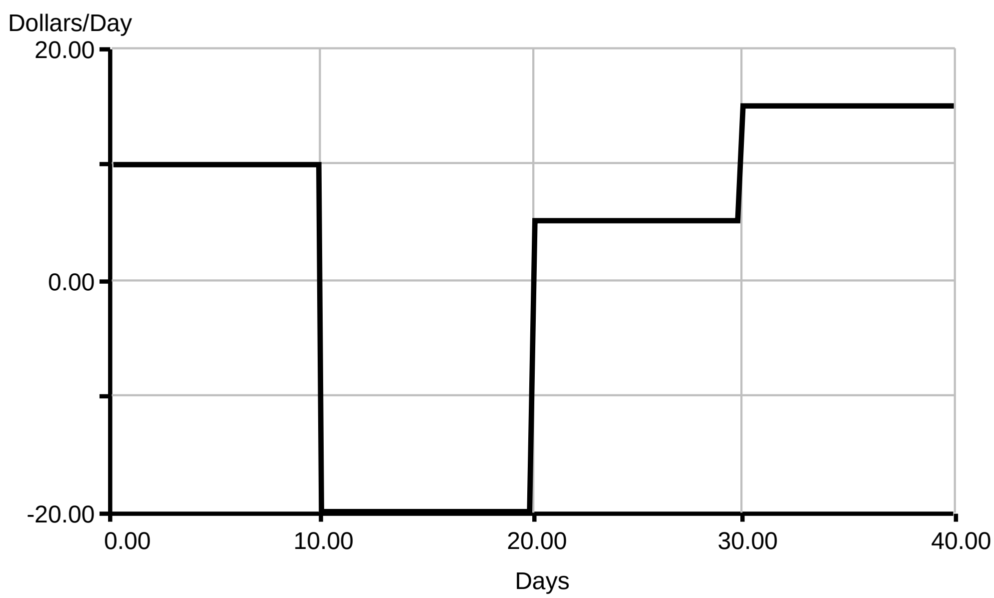


Figure 10: Net Flow of Joe's Bank Account

7.2 Solutions to Exercise 2: Golf Course Development

First, divide the graph into sections as shown in Figure 11. It is important to have enough sections to describe the behavior, but not so many as to over-complicate the calculation process.

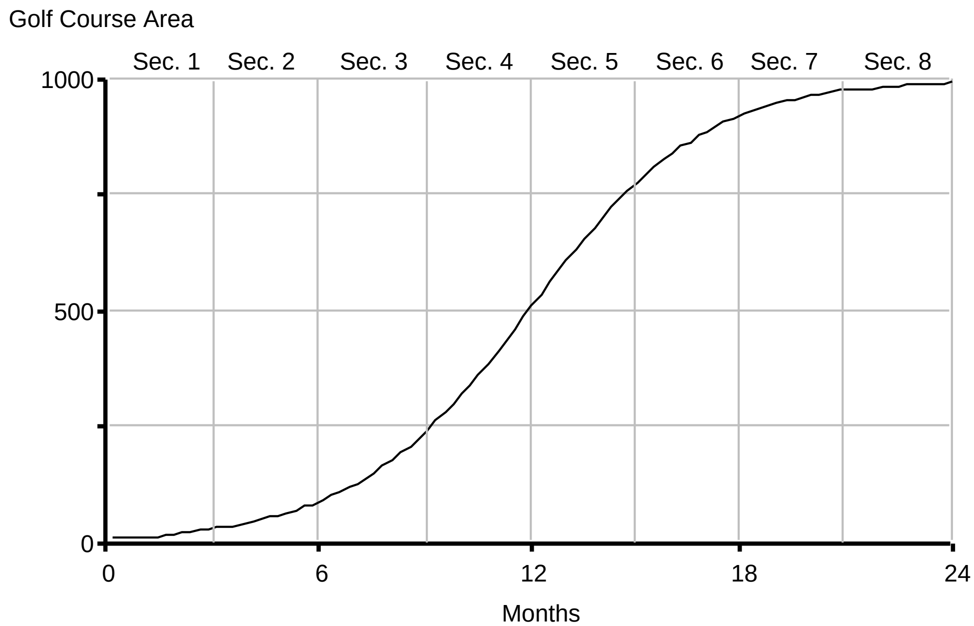


Figure 11: Division of golf course development into sections

Next, calculate the slopes within these sections, shown in Table 4, as in the previous examples.

Table 4: Section based slope calculations for golf course development

Section	Initial Value (acres)	Final Value (acres)	T (months)	Slope (acres/month)
1	10	30	3	7
2	30	90	3	20
3	90	240	3	50
4	240	510	3	90
5	510	780	3	90
6	780	920	3	47
7	920	980	3	20
8	980	990	3	3

Next, pick several points of interest on the graph and draw tangent lines, as in Figure 12. It is usually not necessary to draw as many tangent lines as sections because the tangent lines can be drawn so as to concentrate around the more interesting parts of the graph.

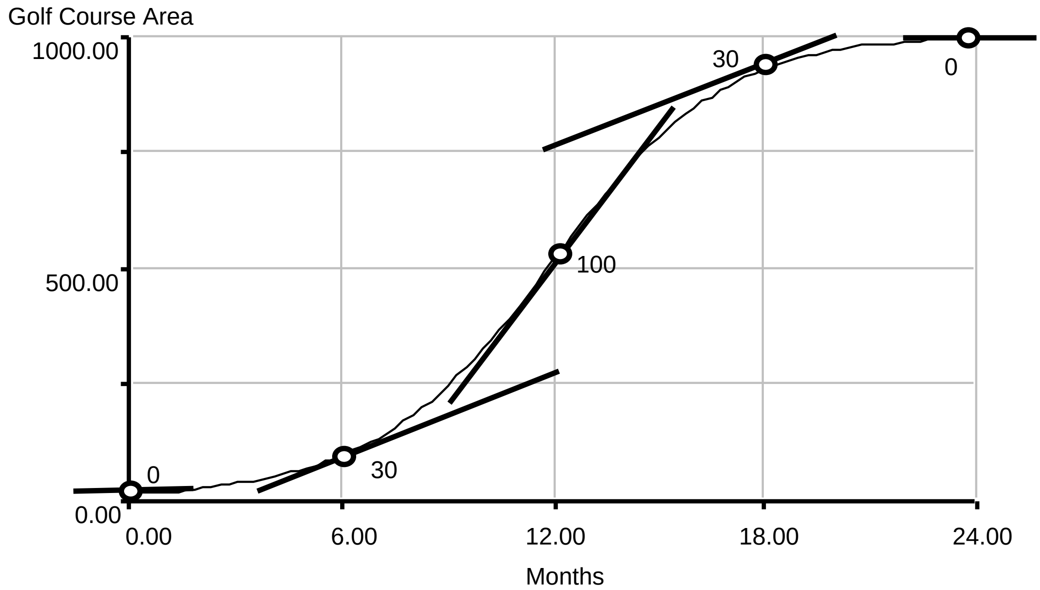


Figure 12: Tangent lines for golf course development

Calculate the slopes of these tangent lines. For example, the fourth tangent line crosses the 12 and 18 month points on the horizontal axis. At the 18 month mark, its value, F, is 940 acres. The value at 12 months, I, is 750 acres. This is over a time period, T, of 6 months. The slope is

$$\frac{F-I}{T} = \frac{190}{6} = 31.\overline{6} \approx 30 \text{ acres/month.}$$

We can round the answer to the nearest 10 because this type of measurement of slope is not good enough to warrant more precision and multiples of 10 are easier numbers to work with. The other calculations are done exactly the same way. When the sectioning results are combined with the slope numbers, a scaled graph can be produced, as shown in Figure 13.

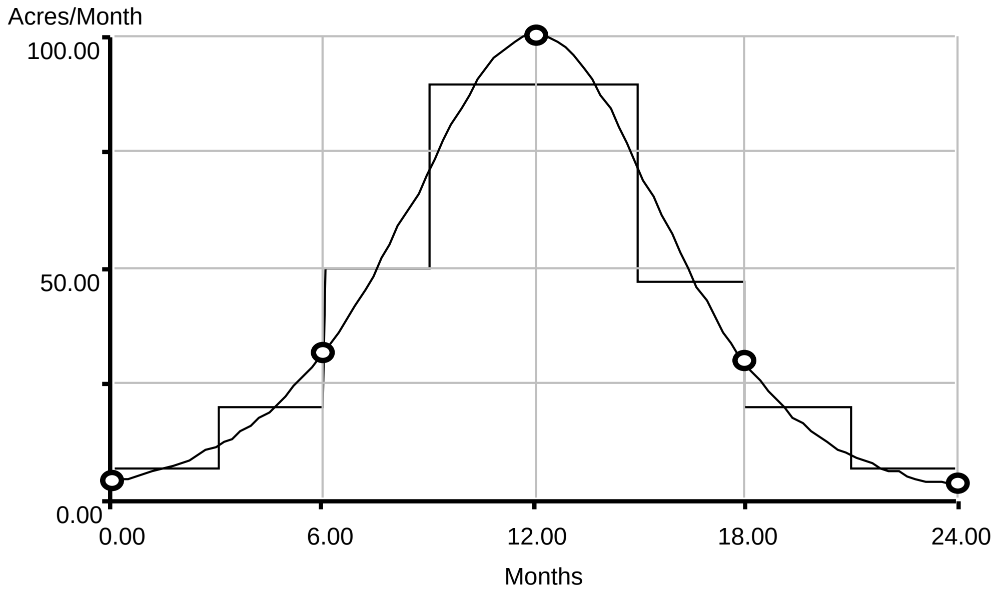


Figure 13: Rate of golf course development fit to section midpoints and tangent line points.

It should be noted that tangent lines have a limited range of accuracy. In this case, it is difficult to be closer than 5 units to the actual values. For instance, even though the tangent slope calculations showed that the slope at the beginning and end was close to zero, the actual slope turned out to be about 3 in each case.

7.3 Solutions to Exercise 3: Overshoot and Collapse

Begin by sectioning the graph and calculating the slopes, as shown in Figure 14 and Table 5.

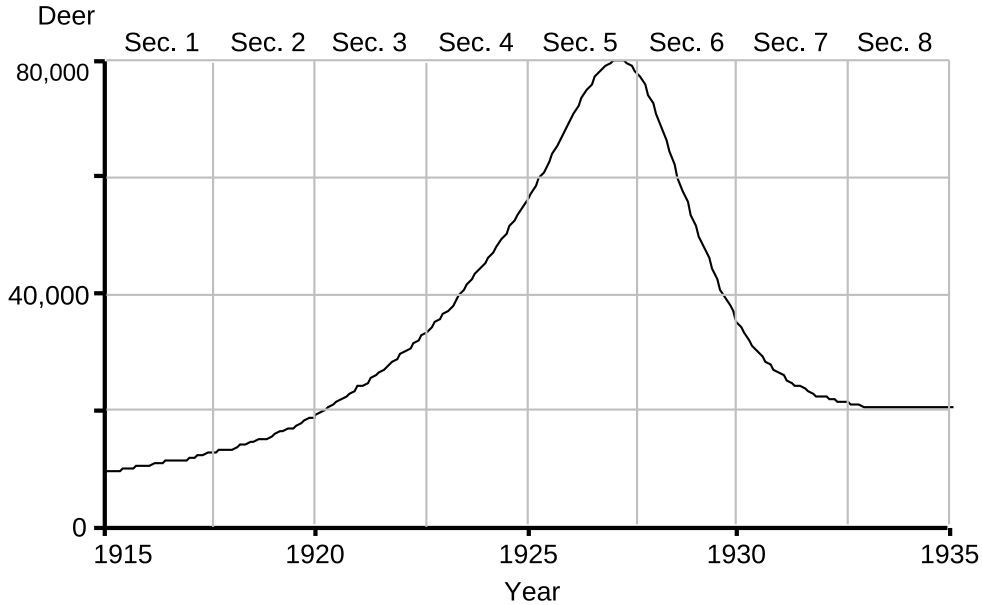


Figure 14: Overshoot and collapse of fishing divided into sections

Table 5: Slopes of the sections of the overshoot and collapse of fishing

Section	Initial Value (Deer)	Final Value (Deer)	T (Years)	Slope (Deer / Year)
1	9200	12,300	2.5	1240
2	12,300	19,100	2.5	2720
3	19,100	33,100	2.5	5600
4	33,100	57,300	2.5	9680
5	57,300	77,600	2.5	8120
6	77,600	34,700	2.5	-17,160
7	34,700	21,500	2.5	-5280
8	21,500	20,600	2.5	-360

One may then draw tangent lines in order to better define the graph, as shown in Figure 15.

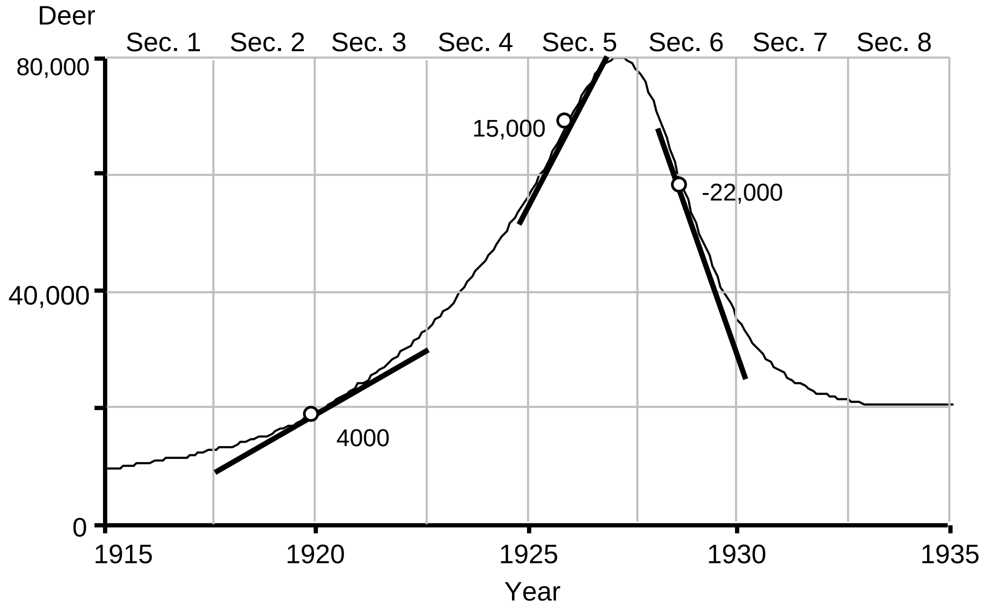


Figure 15: Overshoot and collapse of the deer population with tangent lines

The slopes of these tangent lines may be calculated in the same way as those in Exercise 2. When one combines the sectioning results with the slope information from the tangent lines, one can create a graph such as Figure 16.

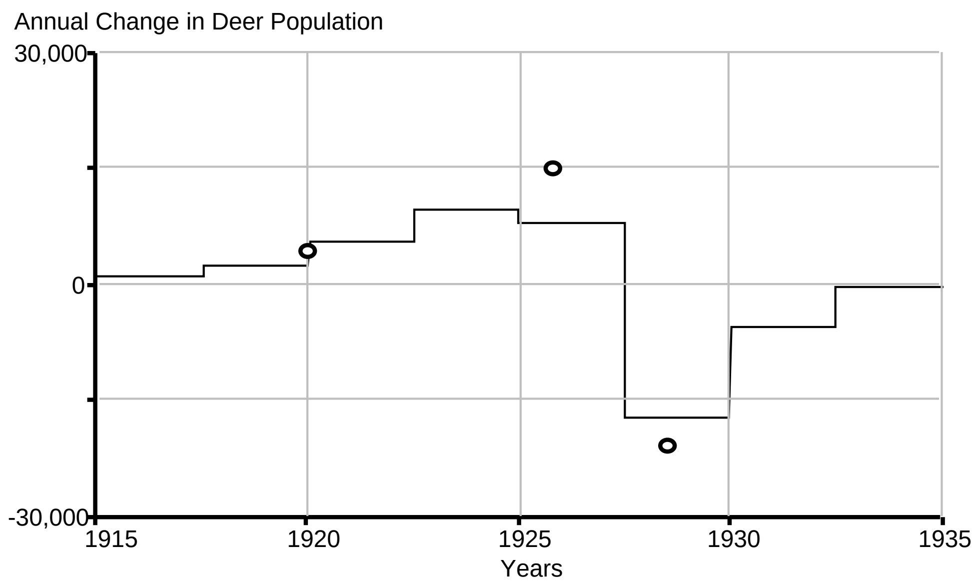


Figure 16: Annual change in deer population with section net slopes and tangent line points

The a net flow line may then be drawn by stretching the shape of the midpoints of the section net slopes to include the tangent line points. The result is shown in Figure 17.

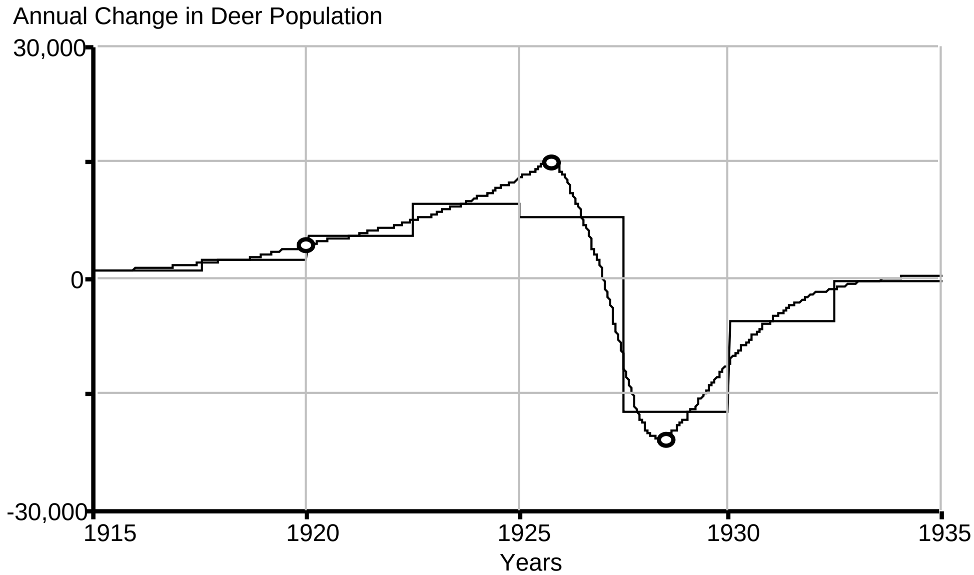


Figure 17: Annual change in deer population fit to section midpoints and tangent line points

It is rare that the net flow of a stock appears more complex than the stock itself, however this is one of those occasions. The net flow of overshoot and collapse is a very complicated behavior and one should not expect to be able to draw it easily. Note that the stock increases as long as the net flow is above zero. Once it drops below zero, the stock begins to decrease in value. Thus the high point of the stock is not in 1926, the high point of the net flow, but in 1927, the year in which the net flow reaches zero.