Last Name	First Name	ID Number

M.I.T.
Sloan School of Management

15.450-Spring 2010
Professor Leonid Kogan

Final Exam

Instructions: Carefully read these instructions! Failure to follow them may lead to deductions from your grade.

1. Immediately write your name at the top of this page.
2. You will turn in only the exam, so write all answers on the exam.
3. Do not in any way communicate with other people taking the exam. Any communication - even if it is not about the exam - will result in a grade of zero.
4. Do not use computers during the exam. Do not use notes or books other than the "cheat sheet" permitted by me.
5. Write your answers clearly and provide explanations.

Problem	Maximum Points	Points
1	15	
2	25	
3	30	
4	30	
Total	100	

1. [15 points] Suppose X_t solves a stochastic differential equation

$$dX_t = \mu X_t \, dt + \sigma X_t (1 - X_t) \, dZ_t$$

Describe $\ln X_t$ as an Ito process, i.e., derive the drift and the diffusion coefficients for $\ln X_t$ as functions of X_t.

2. [25 points] You observe two return series (X_t^1, X_t^2), $t = 1, ..., T$. Assume that these returns are correlated with each other, but independently distributed across time.

(a) Describe a test of the hypothesis that the two return series have the same mean. You are looking for a test with the 5% size. Use large-T asymptotic approximations to construct your test.

(b) Assume now that your sample is relatively short and you are not comfortable relying on large-sample approximations. Describe how you could use the bootstrap methodology to *improve* the accuracy of your tests, i.e., to make sure that the size of your new test is closer to 5% than the size of the test in part (a).

3. [30 points] Consider a market with a riskless bond, paying a constant interest rate r, and a stock paying no dividends with the price process following

$$\begin{aligned} dS_t &= \mu \exp(X_t) S_t \, dt + \sigma S_t \, dZ_t \\ dX_t &= -\theta X_t \, dt + \beta dZ'_t \\ dZ_t dZ'_t &= 0 \end{aligned}$$

Consider a digital option paying \$1 at time T if and only if $S_T \geq K$. K is the strike price of the option.

(a) Characterize the price of the digital option using the risk-neutral approach.

(b) Derive a PDE (with the terminal condition) on the option price.

(c) Using the risk-neutral representation, derive the option price as a function of the stock price and time.

(d) Denote the option price by $D(t, S_t)$. Treating the function $D(t, S_t)$ as known, compute the instantaneous expected excess rate of return on the option at time t given the stock price S_t.

4. [30 points] Your task is to estimate a volatility forecasting model. You observe a time series of observation pairs (X_{t-1}, Y_t), $t = 1, ..., T$. Assume that T is large enough so you can use large-sample asymptotic approximations.

The model states that the conditional volatility of Y_t is a function of X_t:

$$Y_t = \exp\left[\frac{1}{2}(a_0 + a_1 X_{t-1})\right]\varepsilon_t,$$

where ε_t are IID over time, independent of X_{t-1}, and have zero mean: $\mathrm{E}_{t-1}[\varepsilon_t] = 0$, but the exact distribution of ε_t is not known.

Your task is to estimate parameters a_0 and a_1.

(a) Using the QMLE approach, assume that Y_t has a normal distribution

$$Y_t \sim \mathcal{N}(0, \exp(a_0 + a_1 X_{t-1}))$$

Write down the log-likelihood function $\mathcal{L}(a_0, a_1)$.

(b) Using the QMLE approach, derive the two moment conditions that can be used to estimate a_0 and a_1. Show that these are valid moment conditions.

(c) Outline how you could compute standard errors for your parameter estimates.

(d) Describe how you would test the hypothesis that $a_1 = 0$. You need a test with the 5% size. If you were not able to solve the previous part of the question, for this part you can assume that you know the variance-covariance matrix of $(\widehat{a}_0, \widehat{a}_1)$, denoted by $\widehat{V}$.

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