

Problem Set 1

1. (To be solved individually) Consider a one-period model of the market. Assume there are three possible states at time $t = 1$: 0, 1, and 2, all equally likely. There are two assets traded in this market: the stock and the risk-free bond. Assume that at time $t = 1$ the bond pays \$1 in each state, and stock pays \$0, \$1, and \$2 in states 0, 1, and 2 respectively. The time-0 price of the stock is \$0.8, and the initial price of the bond is \$0.9. When dealing with the state-price densities, always normalize them to be 1 at time 0.
 - (a) Show that the above model is free from arbitrage. (Hint: demonstrate that there exists at least one state-price density in this model).
 - (b) How many state-price densities exist in this model? Characterize all of them.
 - (c) Suppose that an option is introduced into the above market, which pays $\max(S_1 - 1, 0)$ at time 1. Using payoff dominance arguments, show that the time-0 price of the option is bounded from below by \$0, and from above by \$0.4. (Hint: try to bound the payoff of the option by the payoff of a portfolio including the stock and the bond).
 - (d) Compute the time-0 price of the option under each of the state-price densities you have derived in item (1b). What is the lowest and the highest possible price of the option consistent with absence of arbitrage? Compare with the answer in item (1c).
 - (e) Suppose that the option in item (1c) trades at \$0.2 at time 0. Show that the SPD is unique and derive it.
 - (f) Continuing with the assumptions in the item (1e), suppose that there is one more option available for trading, paying $\max(S_1 - 0.5, 0)$ at time 1. Compute the time-0 price of this option. Argue why there is a unique price consistent with no arbitrage.
 - (g) Replicate the payoff of the option paying $\max(S_1 - 0.5, 0)$ at time 1 using a portfolio of the stock, the bond, and the option paying $\max(S_1 - 1, 0)$ at time 1.
2. (To be solved in a group) Consider an extension of the Black-Scholes model. Assume that the gross return on the stock between $t = 0$ and $t = 1$ is given by

$$R_T = \exp(\mu + \sigma\varepsilon_1 - \nu\xi_1),$$

where ε_1 and ξ_1 are independent. Assume that $\varepsilon_1 \sim \mathcal{N}(0, 1)$, while ξ_1 is exponentially distributed: for any a , $\text{Prob}(\xi_1 > a) = \exp(-a)$. Assume that $\nu > 0$. Assume that the gross return on the risk-free bond between 0 and 1 is given by $\exp(r)$.

The above model adds a single negative jump to the Black-Scholes setting. ν parameterizes the distribution of the jump.

Assume the following parameter values:

$$r = 0.05, \sigma = 0.2, \nu = 0.05.$$

Assume that under the risk-neutral probability, the jump remains exponentially distributed: the risk-neutral distribution of ξ_1 is the same as physical, while the risk-neutral jump distribution parameter is $\nu^{\mathbf{Q}} = 0.2$. Moreover, assume that the risk-neutral distribution of ε_1 is normal with variance 1 and mean $-\eta$, $\eta = 0.25$.

- (a) Compute μ (Hint: you may want to look up the moment-generating function for an exponential random variable).
 - (b) Assume that the initial stock price is $S_0 = 1$. Consider European put options maturing at $T = 1$ with strike prices $K = 0.5, 0.6, \dots, 1, \dots, 1.5$. Compute the Black-Scholes implied volatilities for these put options and plot them as a function of the strike price. (Hint: Conditional on ξ_1 , stock return distribution is lognormal, like in the Black-Scholes model). Compute the absolute value of the Sharpe ratio of returns on each put option. Repeat this exercise for $\nu^{\mathbf{Q}} = 0.05$. Comment on the differences in results.
3. (To be solved individually) Consider a zero-coupon corporate bond maturing at time $t = 1$, with the face value of \$1. Assume that the state of the firm is captured by the random variable X_1 . The bond defaults if $X_1 \leq A$, in which case it pays nothing. Assume that $X_1 \sim \mathcal{N}(0, 1)$. Let the state-price density be given by

$$\pi_1 = \exp(-r_f - \eta^2/2 - \eta Y_1), \quad \pi_0 = 1$$

where $Y_1 \sim \mathcal{N}(0, 1)$. Assume that $\text{corr}(X_1, Y_1) = \rho$. (Hint: Y_1 can be written as $\rho X_1 + \sqrt{(1 - \rho^2)} Z_1$ where X_1 and Z_1 are independent).

You are supposed to perform the following analysis analytically. To help your intuition, you may want to perform numerical calculations first by assuming specific parameter values.

- (a) Derive the default probability p_{def} .
- (b) Derive the price of the bond at time $t = 0$. (Hint: It is simpler to perform calculations under the risk-neutral probability measure).
- (c) Derive the bond yield and the Sharpe ratio of bond returns. Relate both to ρ , η , p_{def} , and comment on the relationship.

- (d) This question is relatively open-ended. A qualitative answer would be acceptable. How would you build a model of corporate bond prices, so that bond yields change significantly over time (e.g., widen in times of economic distress), while the likelihood of default changes much less?

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15.450 Analytics of Finance
Fall 2010

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