# 15.401 Finance Theory

*MIT Sloan MBA Program*

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## *Lectures 10–11: Options*

# Critical Concepts

- Motivation
- Payoff Diagrams
- Payoff Tables
- Option Strategies
- Other Option-Like Securities
- Valuation of Options
- A Brief History of Option-Pricing Theory
- Extensions and Qualifications

## Readings:

- Brealey, Myers, and Allen Chapters 27

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# Motivation

**Derivative: Claim Whose Payoff is a Function of Another's**

- **Warrants**, **Options**
- **Calls** vs. **Puts**
- **American** vs. **European**
- Terms: **Strike Price** $K$, **Time to Maturity** $\tau$

$$S_t \ = \ \text{Stock Price} \ , \ K \ = \ \text{Strike Price}$$
$$C_t \ = \ \text{Call Price} \quad , \ P_t \ = \ \text{Put Price}$$

- At Maturity $T$, payoff

$$C_T \ = \ \text{Max} \ [ \ 0 \ , \ S_T - K \ ]$$
$$P_T \ = \ \text{Max} \ [ \ 0 \ , \ K - S_T \ ]$$

# Motivation

**Put Options As Insurance**

$$\text{Asset Insured} = \text{Stock}$$
$$\text{Current Asset Value} =$$
$$\text{Term of Policy} =$$
$$\text{Maximum Coverage} =$$
$$\text{Deductible} =$$
$$\text{Insurance Premium} =$$

- Differences
  - Early exercise
  - Marketability
  - Dividends

# Payoff Diagrams

## Call Option:

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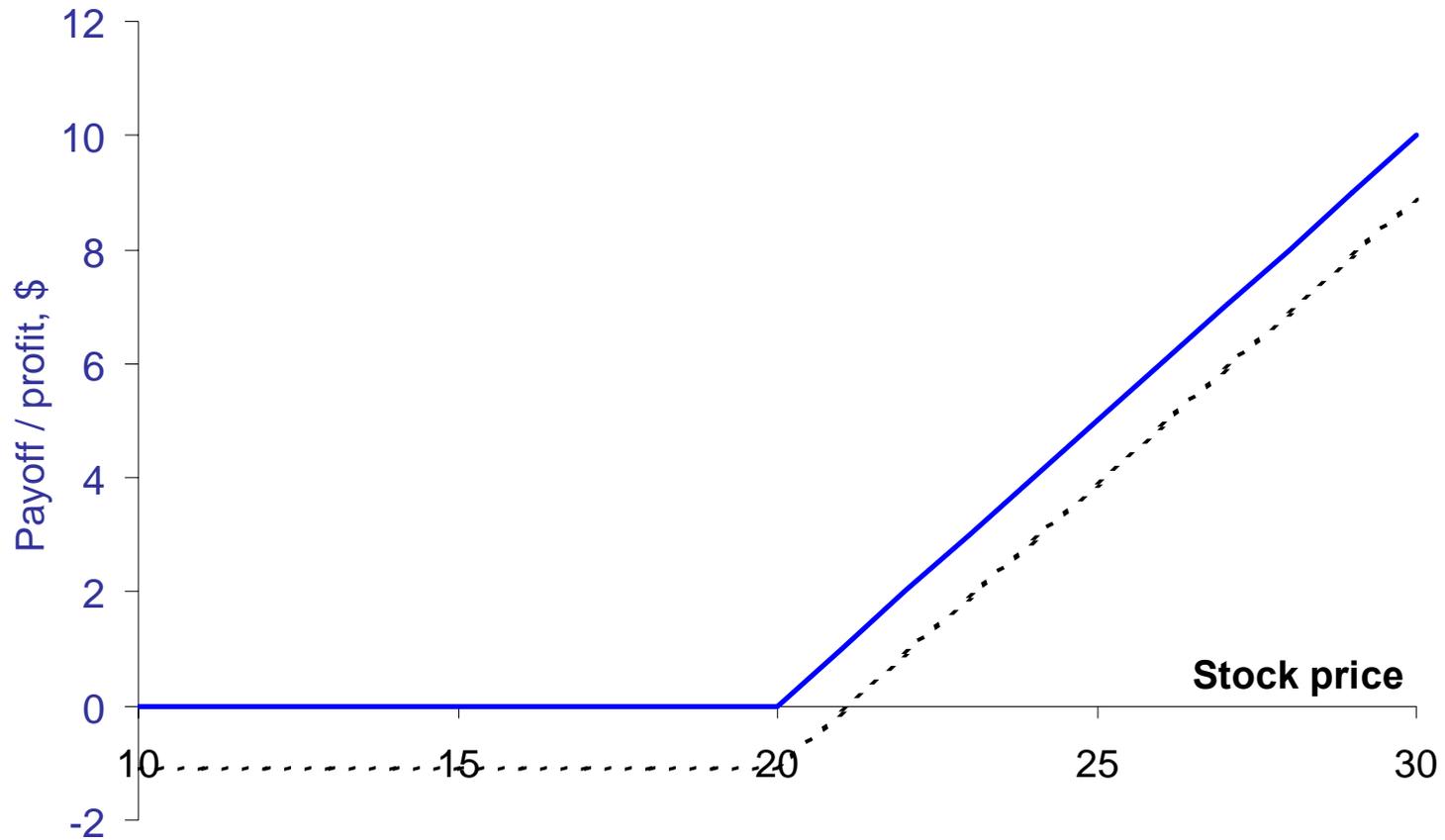
## Stock <u>Return</u> vs. Call Option <u>Return</u>
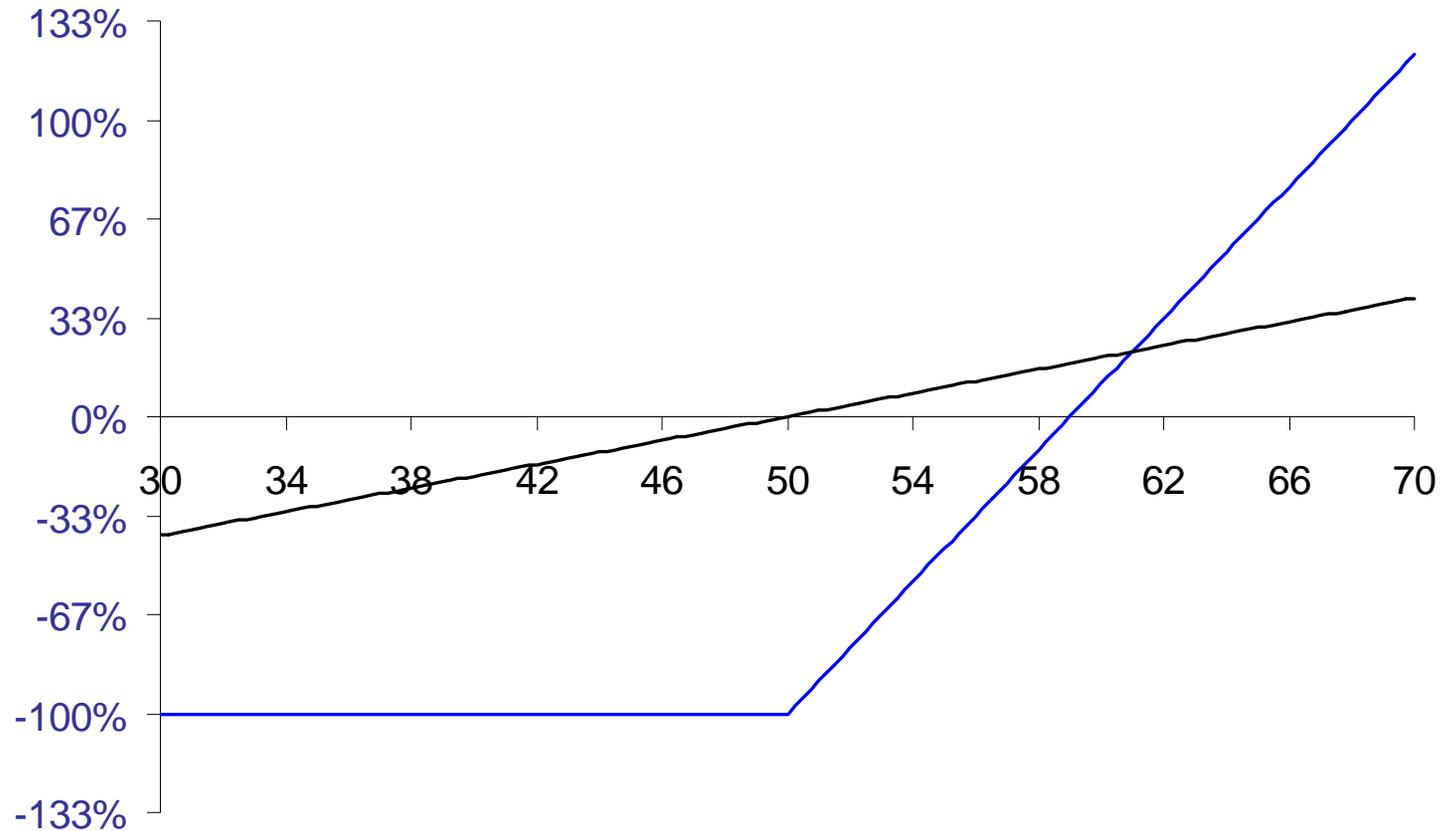
# Payoff Diagrams

## Put Option:

# Payoff Diagrams

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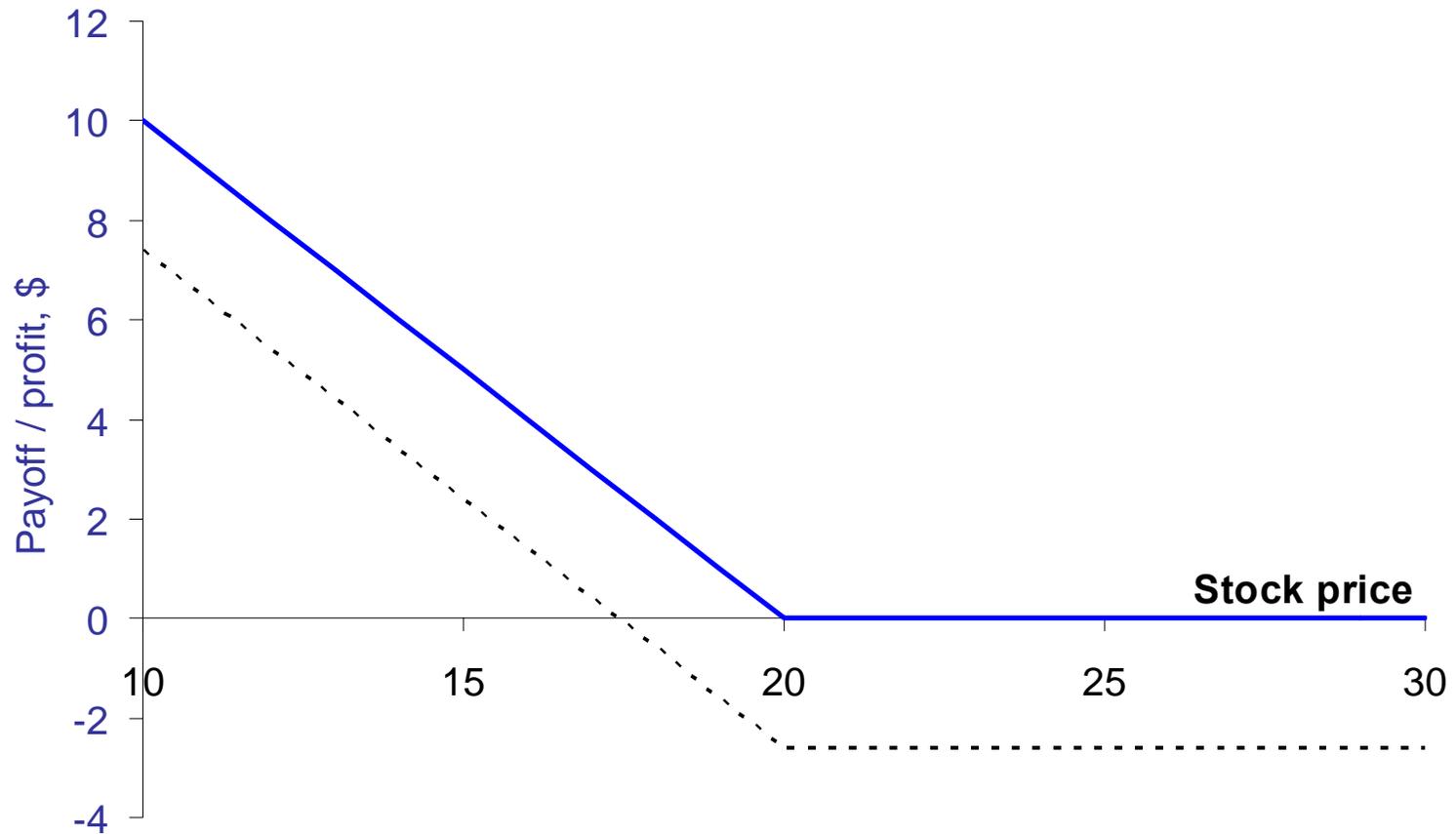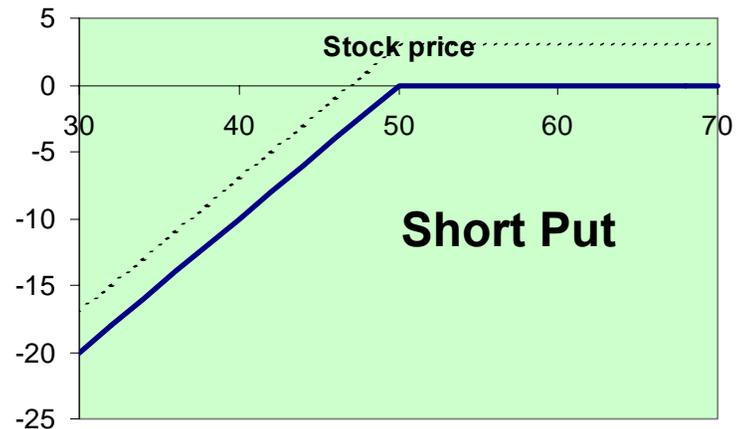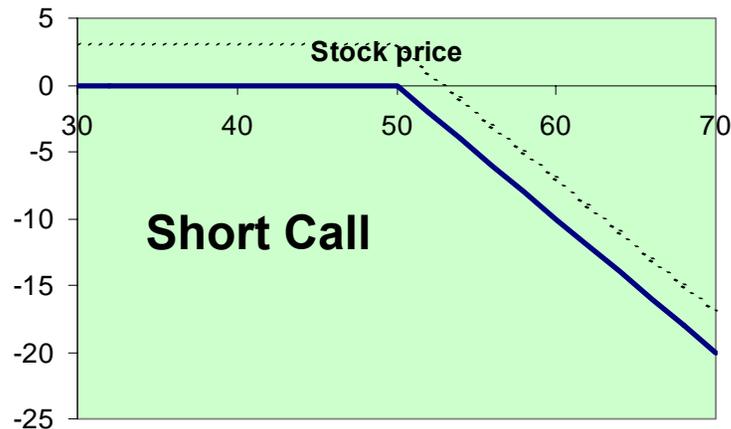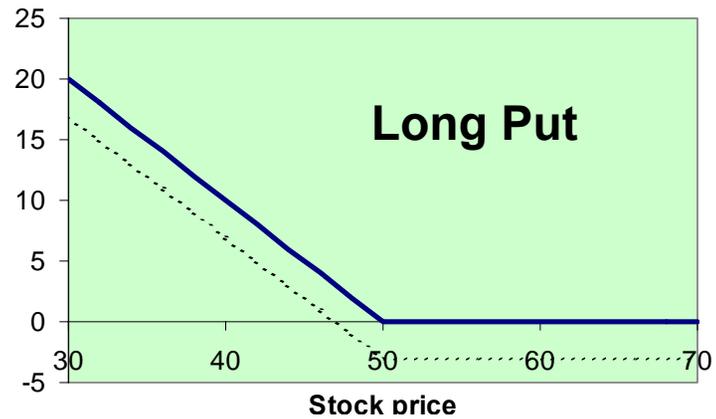
# Payoff Tables

**Stock Price = S, Strike Price = X**

### Call option (price = C)

|         | if S < X | if S = X | if S > X    |
|---------|----------|----------|-------------|
| Payoff  | 0        | 0        | S – X       |
| Profit  | –C       | –C       | S – X – C   |

### Put option (price = P)

|         | if S < X   | if S = X | if S > X |
|---------|------------|----------|----------|
| Payoff  | X – S      | 0        | 0        |
| Profit  | X – S – P  | –P       | –P       |

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# Option Strategies

**Trading strategies**

Options can be combined in various ways to create an unlimited number of payoff profiles.

**Examples**

- Buy a stock and a put
- Buy a call with one strike price and sell a call with another
- Buy a call and a put with the same strike price

# Option Strategies

## Stock + Put



Buy stock

Buy a put

Stock + put

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## Call1 ~ Call2

# Option Strategies

## Call + Put

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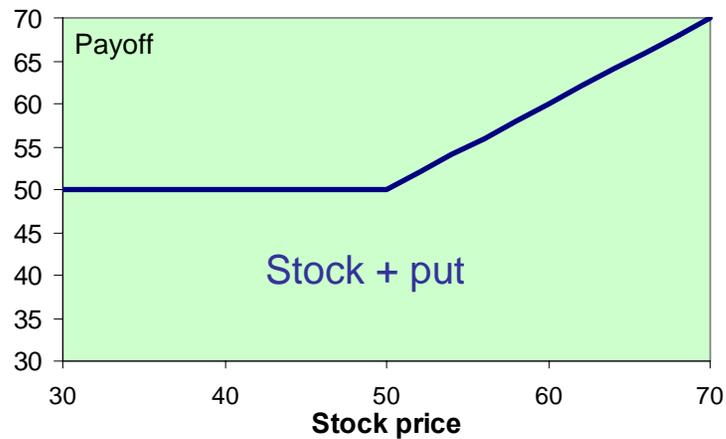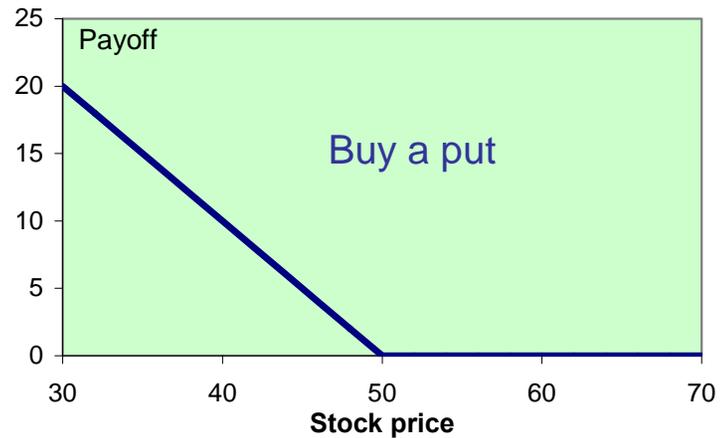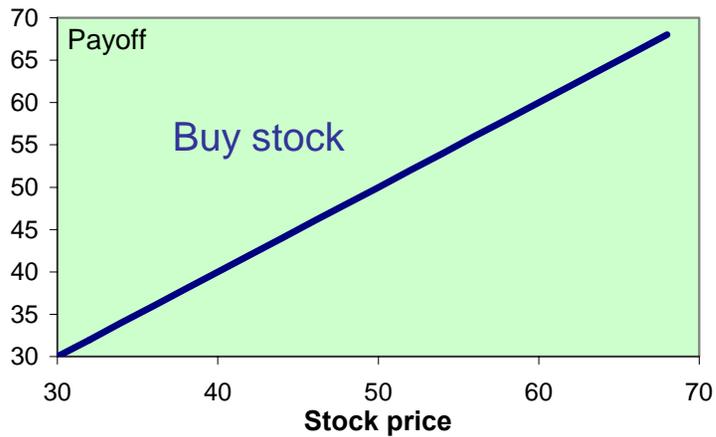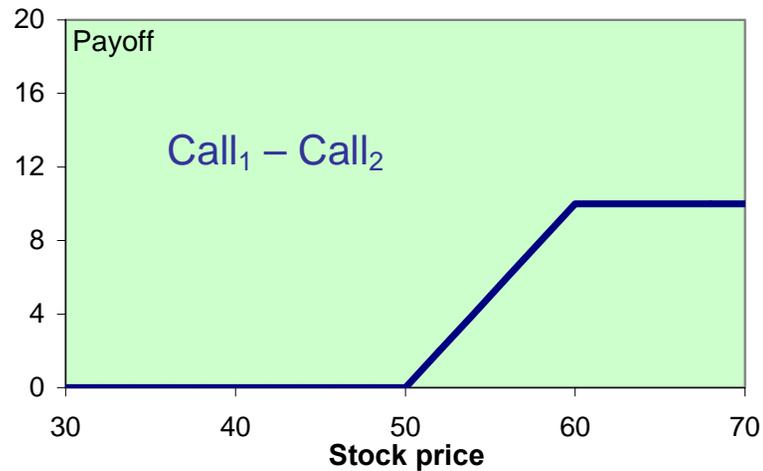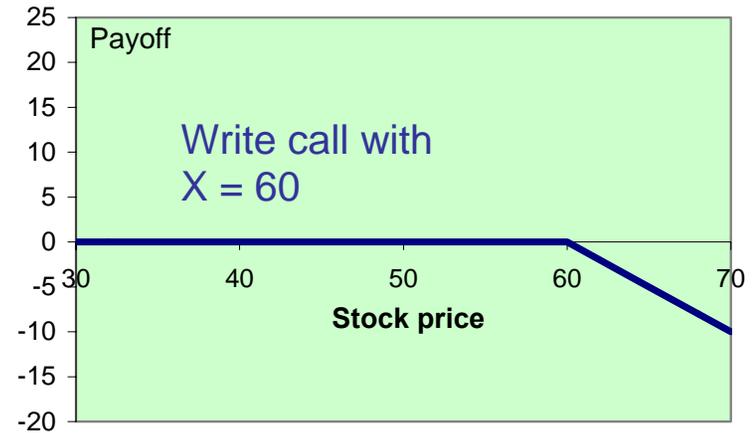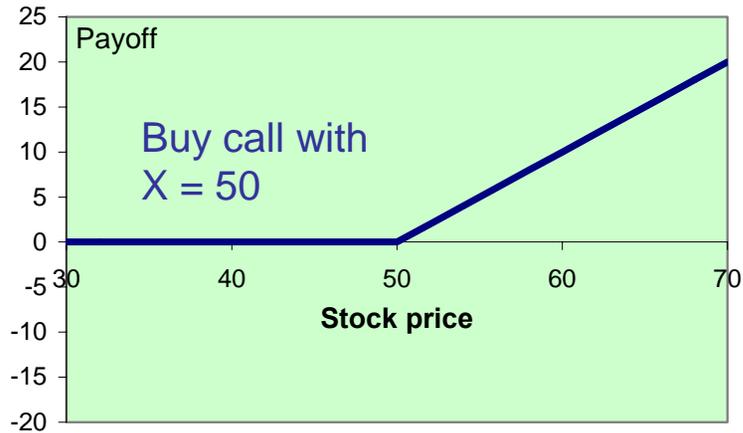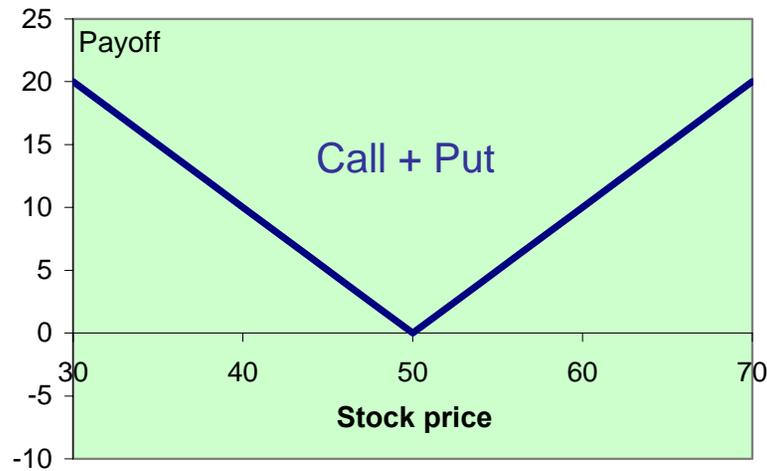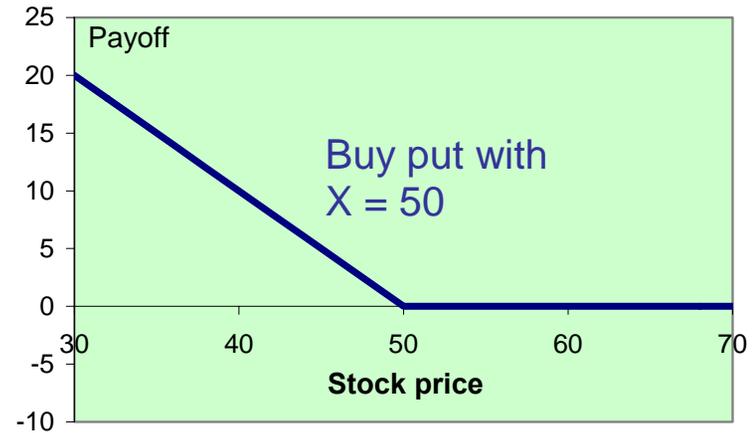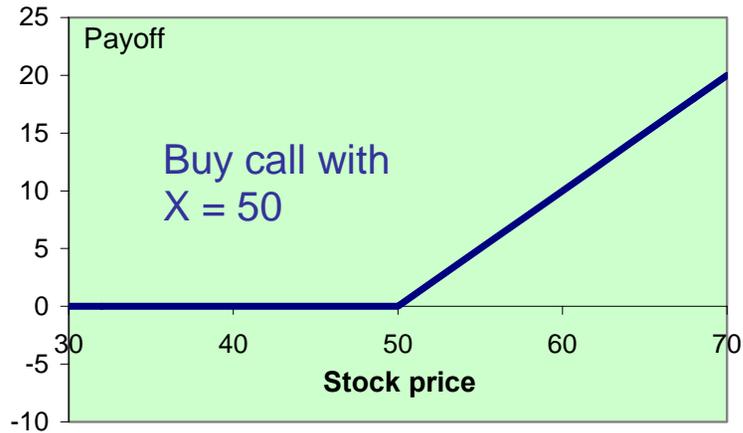
## Corporate Liabilities:

$$\text{Equity} \equiv \text{Max}\,[\,0\,,\,V{-}B\,] \equiv E$$
$$= V \,-\, B \,+\, \text{Max}\,[\,0\,,\,B{-}V\,]$$

$$\text{Debt} \equiv \text{Min}\,[\,V\,,\,B\,] \equiv D$$
$$= B \,-\, \text{Max}\,[\,0\,,\,B{-}V\,]$$

$$V \,=\, D \,+\, E$$

- Equity:  hold call, or protected levered put
- Debt: issue put and (riskless) lending
- Quantifies Compensation Incentive Problems:
  - Biased towards higher risk
  - Can overwhelm NPV considerations
  - Example: leveraged equity

# Other Option-Like Securities

**Other Examples of Derivative Securities:**

- Asian or "Look Back" Options
- Callables, Convertibles
- Futures, Forwards
- Swaps, Caps, Floors
- Real Investment Opportunities
- Patents
- Tenure
- etc.

**Binomial Option-Pricing Model of Cox, Ross, and Rubinstein (1979):**

- Consider One-Period Call Option On Stock XYZ
- Current Stock Price $S_0$
- Strike Price $K$
- Option Expires Tomorrow, $C_1 = \text{Max}[S_1 - K, 0]$
- What Is Today's Option Price $C_0$?

$$0 \qquad\qquad\qquad\qquad\qquad\qquad 1$$

$$\vdash\!\!-\!\!-\!\!-\!\!-\!\!-\!\!-\!\!-\!\!-\!\!-\!\!-\!\!-\!\!-\!\!-\!\!-\!\!-\!\dashv$$

$$S_0 , C_0 = \text{???} \qquad\qquad\qquad S_1 , C_1 = \text{Max}[S_1 - K, 0]$$

# Valuation of Options

**Suppose $S_0 \rightarrow S_1$ Is A Bernoulli Trial:**

$$S_1 \begin{cases} \xrightarrow{p} uS_0 \\ \\ \xrightarrow{1-p} dS_0 \end{cases} \qquad u > d$$

$$C_1 \begin{cases} \xrightarrow{p} \mathrm{Max}\,[\,uS_0 - K\,,\,0\,] == C_u \\ \\ \xrightarrow{1-p} \mathrm{Max}\,[\,dS_0 - K\,,\,0\,] == C_d \end{cases}$$

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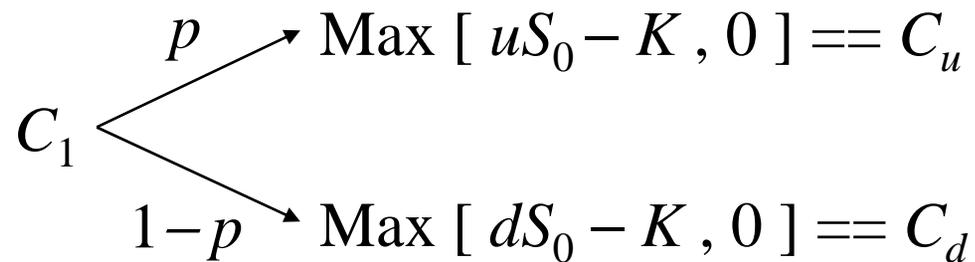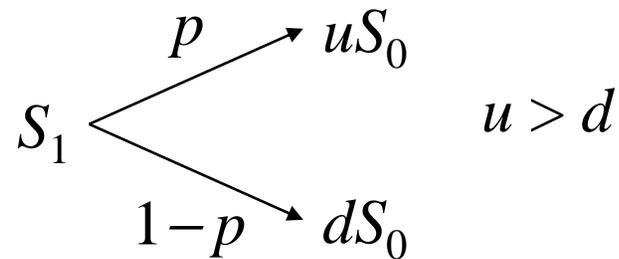
**Now What Should $C_0 = f(\cdot)$ Depend On?**

- Parameters: $S_0$, $K$, $u$, $d$, $p$, $r$

**Consider Portfolio of Stocks and Bonds Today:**

- $\Delta$ Shares of Stock, $\$B$ of Bonds
- Total Cost Today: $V_0 == S_0\Delta + B$
- Payoff $V_1$ Tomorrow:

$$V_1 \begin{array}{c} \xrightarrow{p} uS_0\Delta + rB \\ \\ \xrightarrow{1-p} dS_0\Delta + rB \end{array}$$
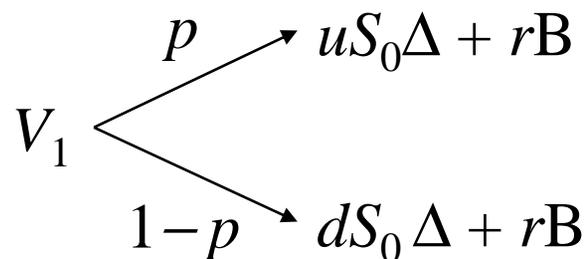
**Now Choose $\Delta$ and *B* So That:**

$$V_1 \underset{1-p}{\overset{p}{\diagdown}} \begin{array}{l} uS_0\Delta + r\mathrm{B} = C_u \\ \\ dS_0\Delta + r\mathrm{B} = C_d \end{array} \Bigg\} \Rightarrow \begin{array}{l} \Delta^* = (C_u - C_d)/(u-d)S_0 \\ \\ B^* = (uC_d - dC_u)/(u-d)r \end{array}$$

**Then It Must Follow That:**

$$C_0 = V_0 = S_0\Delta^* + B^* = \frac{1}{r}\left[\frac{r-d}{u-d}\ C_u + \frac{u-r}{u-d}\ C_d\right]$$

# Valuation of Options

**Suppose $C_0 > V_0$**

- Today: Sell $C_0$ , Buy $V_0$ , Receive $C_0 - V_0 > 0$
- Tomorrow: Owe $C_1$, But $V_1$ Is Equal To $C_1$!

**Suppose $C_0 < V_0$**

- Today: Buy $C_0$ , Sell $V_0$ , Receive $V_0 - C_0 > 0$
- Tomorrow: Owe $V_1$, But $C_1$ Is Equal To $V_1$!

**$C_0 = V_0$ , Otherwise Arbitrage Exists**

$$C_0 = V_0 = S_0 \Delta^* + B^* = \frac{1}{r} \left[ \frac{r - d}{u - d} C_u + \frac{u - r}{u - d} C_d \right]$$

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# Valuation of Options

## Note That *p* Does Not Appear Anywhere!

- Can disagree on *p*, but must agree on option price

- If price violates this relation, arbitrage!

- A multi-period generalization exists:

$$C_0 \;=\; \frac{1}{r^n} \sum_{k=0}^{n} \binom{n}{k} p^{*k}(1-p^*)^{n-k} \text{Max}[0, u^k d^{n-k} S_0 - K]$$

$$p^* \;\equiv\; \frac{r-d}{u-d}$$

- Continuous-time/continuous-state version (Black-Scholes/Merton):

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} \;+\; rS\frac{\partial C}{\partial S} \;+\; \frac{\partial C}{\partial t} \;=\; rC$$

$$C(S,T) \;=\; \text{Max}\,[\,S-K, 0\,]$$

$$C(0,t) \;=\; 0$$

## Earliest Model of Asset Prices:

- Notion of "fair" game

- The **martingale**

- Cardano's (c.1565) *Liber De Ludo Aleae*:

  The most fundamental principle of all in gambling is simply equal conditions, e.g. of opponents, of bystanders, of money, of situation, of the dice box, and of the die itself.  To the extent to which you depart from that equality, if it is in your opponent's favor, you are a fool, and if in your own, you are unjust.

- Precursor of the **Random Walk Hypothesis**

# A Brief History of Option-Pricing Theory

**Louis Bachelier (1870−1946):**

- *Théorie de la Spéculation* (1900)

- First mathematical model of Brownian motion

- Modelled warrant prices on Paris Bourse

$$\frac{1}{c}\frac{\partial \mathcal{P}}{\partial t} - \frac{\partial^2 \mathcal{P}}{\partial x^2} = 0$$

- Poincaré's evaluation of Bachelier:

    The manner in which the candidate obtains the law of Gauss is most original, and all the more interesting as the same reasoning might, with a few changes,be extended to the theory of errors. He develops this in a chapter which might at first seem strange, for he titles it ``Radiation of Probability''. In effect, the author resorts to a comparison with the analytical theory of the propagation of heat. A little reflection shows that the analogy is real and the comparison legitimate. Fourier's reasoning is applicable almost without change to this problem, which is so different from that for which it had been created. It is regrettable that [the author] did not develop this part of his thesis further.

# A Brief History of Option-Pricing Theory

**Kruizenga (1956):**

- MIT Ph.D. Student (Samuelson)

- "Put and Call Options: A Theoretical and Market Analysis"

**Sprenkle (1961):**

- Yale Ph.D. student (Okun, Tobin)

- "Warrant Prices As Indicators of Expectations and Preferences"

**What Is The "Value" of Max [ $S_T - K$, 0 ]?**

- Assume probability law for $S_T$

- Calculate $E_t$ [ Max [ $S_T - K$, 0 ] ]

- But what is its _price_?

- Do risk preferences matter?

# A Brief History of Option-Pricing Theory

## Samuelson (1965):

- "Rational Theory of Warrant Pricing"

- "Appendix: A Free Boundary Problem for the Heat Equation Arising from a Problem in Mathematical Economics",  H.P. McKean

## Samuelson and Merton (1969):

- "A Complete Model of Warrant Pricing that Maximizes Utility"

- Uses preferences to value expected payoff

## Black and Scholes (1973):

- Construct portfolio of options and stock

- Eliminate market risk (appeal to CAPM)

- Remaining risk is inconsequential (idiosyncratic)

- Expected return given by CAPM (PDE)

# A Brief History of Option-Pricing Theory

**Merton (1973):**

- Use continuous-time framework
  - Construct portfolio of options and stock
  - Eliminate market risk (dynamically)
  - Eliminate idiosyncratic risk (dynamically)
  - Zero payoff at maturity
  - No risk, zero payoff $\Rightarrow$ zero cost (otherwise arbitrage)
  - Same PDE
- More importantly, a "production process" for derivatives!
- Formed the basis of **financial engineering** and three distinct industries:
  - Listed options
  - OTC structured products
  - Credit derivatives

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## The Rest Is Financial History:

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# Extensions and Qualifications

- The derivatives pricing literature is HUGE

- The mathematical tools involved can be quite challenging

- Recent areas of innovation include:

  - Empirical methods in estimating option pricings

  - Non-parametric option-pricing models

  - Models of credit derivatives

  - Structured products with hedge funds as the underlying

  - New methods for managing the risks of derivatives

  - Quasi-Monte-Carlo methods for simulation estimators

- Computational demands can be quite significant

- This is considered "rocket science"

# Key Points

- Options have nonlinear payoffs, as diagrams show
- Some options can be viewed as insurance contracts
- Option strategies allow investors to take more sophisticated bets
- Valuation is typically derived via arbitrage arguments (e.g., binomial)
- Option-pricing models have a long and illustrious history

# Additional References

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