Enumerative Methods

A knapsack problem

- Let's focus on maximization integer linear programs with only binary variables
- For example: a knapsack problem with 6 items

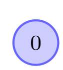
$$\begin{aligned} \max \quad & 16x_1 + 22x_2 + 12x_3 + 8x_4 + 11x_5 + 19x_6 \\ \text{s.t.} \quad & 5x_1 + 7x_2 + 4x_3 + 3x_4 + 4x_5 + 6x_6 \le 14 \\ & x_1, x_2, \ldots, x_6 \in \{0, 1\} \end{aligned}$$

Complete enumeration

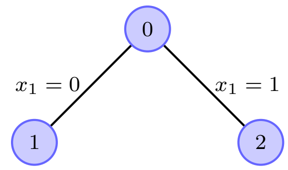
- Complete enumeration systematically considers all possible solutions
  - n binary variables $x_1, \ldots, x_n \Rightarrow 2^n$ possible solutions
- After considering all possible solutions, choose best feasible solution
- Usual idea: iteratively break the problem into 2
  - For example, first, we consider consider separately the cases that $x_1 = 0$ and $x_1 = 1$

An enumeration tree

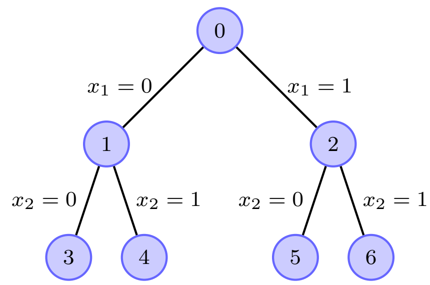
- Let's enumerate all possible solutions of our illustrative knapsack problem
- 6 binary decision variables $x_1, \ldots, x_6$
- We can enumerate all possible solutions systematically using a tree
- Start with **root node**
  - No variables have been fixed in value



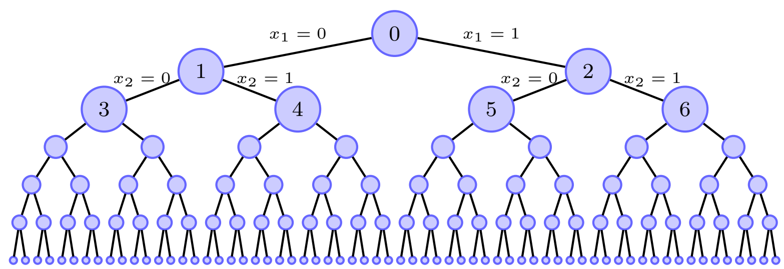
- **Branch** the possibilities for x_1: $x_1 = 0$ or $x_1 = 1$



- Next, branch the possibilities for x_2: $x_2 = 0$ or $x_2 = 1$



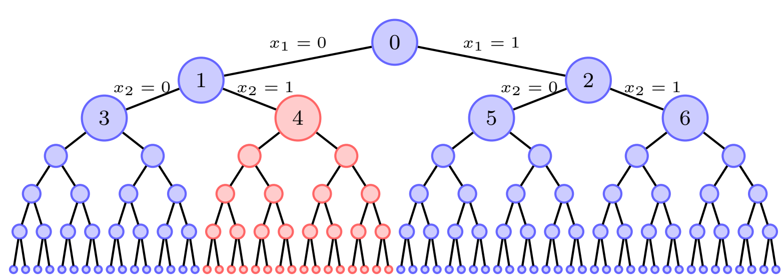
- Keep building the tree, branching the possibilities for x_3, x_4, x_5, x_6



- Each node corresponds to a **partial solution**
  - For example, node 4 $\Leftrightarrow$ fix $x_1 = 0$ and $x_2 = 1$
  - partial solution of node 4 = $\mathbf{x}^{(4)} = (0, 1, \#, \#, \#, \#)$

- Each of the 64 **leaves** of the tree (nodes at the bottom) corresponds to a solution: a complete assignment of variables

Subtrees of an enumeration tree



- **Subtree** (or **descendants**) of node i = nodes obtained from node i from subsequent branching

- Example: red nodes = subtree of node 4
- Recall: node 4 $\Leftrightarrow$ partial solution $\mathbf{x}^{(4)} = (0, 1, \#, \#, \#, \#)$
- Leaves of subtree of node 4 $\Leftrightarrow$ **completions** of $\mathbf{x}^{(4)}$
  - (full) solutions that have the same fixed variables as $\mathbf{x}^{(4)}$
- Idea: stop branching from a node as soon as possible
  - Suppose we look at node 4 and conclude none of its descendants can be optimal

  $\Rightarrow$ Can eliminate 1/4 the solutions at once!

Incumbent solutions

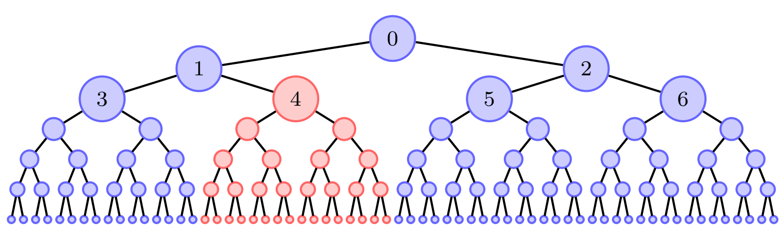
- Goal of branch and bound: find an optimal (or at least a good feasible) solution to some optimization model
- The **incumbent solution** at any stage of branch and bound is the best feasible solution known so far (in terms of objective value)
- Notation:
  - Incumbent solution $\hat{\mathbf{x}}$
  - Incumbent solution's objective function value $\hat{v}$
- Most branch and bound algorithms have subroutines that run at the beginning trying to get a good feasible solution

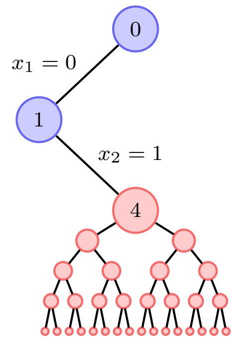
Eliminating nodes and subtrees

- Let's look at our knapsack problem
- Suppose that we have an incumbent solution $\hat{\mathbf{x}}$ with objective value $\hat{v}$:

$$\hat{\mathbf{x}} = (1, 1, 0, 0, 0, 0) \qquad \hat{v} = 38$$

- Let's look at the subtree of node 4 in our enumeration tree





- Node 4 $\Leftrightarrow$ partial solution $\mathbf{x}^{(4)} = (0, 1, \#, \#, \#, \#)$
- All possible completions of $\mathbf{x}^{(4)}$ $\Leftrightarrow$ Leaves of node 4's subtree
- **Candidate problem** for node 4: find the best possible completion of $\mathbf{x}^{(4)}$

$$\begin{aligned} v^{(4)} = \max \quad & 16x_1 + 22x_2 + 12x_3 + 8x_4 + 11x_5 + 19x_6 \\ \text{s.t.} \quad & 5x_1 + 7x_2 + 4x_3 + 3x_4 + 4x_5 + 6x_6 \leq 14 \\ & x_1 = 0, x_2 = 1 \\ & x_1, x_2, \ldots, x_6 \in \{0, 1\} \end{aligned}$$

- LP relaxation gives us <u>upper bound</u> on $v^{(4)}$:

$$\begin{aligned} \tilde{v}^{(4)} = \max \quad & 16x_1 + 22x_2 + 12x_3 + 8x_4 + 11x_5 + 19x_6 \\ \text{s.t.} \quad & 5x_1 + 7x_2 + 4x_3 + 3x_4 + 4x_5 + 6x_6 \leq 14 \\ & x_1 = 0, x_2 = 1 \\ & 0 \leq x_i \leq 1, \quad i = 1, \ldots, 6 \end{aligned}$$

- Solve LP relaxation: $\tilde{v}^{(4)} = 44$
- Best completion of $\mathbf{x}^{(4)}$ has value $v^{(4)} \leq \tilde{v}^{(4)} = 44$
- Incumbent solution has value $\hat{v} = 38$

$\Rightarrow$ It is possible that some completion of $\mathbf{x}^{(4)}$ has a better solution value than 38

$\Rightarrow$ Need to examine solutions that branch from node 4

- What if we had an incumbent solution with value $\hat{v} = 45$?
- Then no completion of $\mathbf{x}^{(4)}$ is better than our incumbent, since

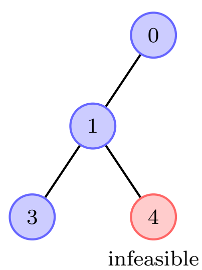
$$v^{(4)} \leq \tilde{v}^{(4)} = 44 < 45 = \hat{v}$$

- We can **terminate** or **fathom** node 4: we do not need to branch the subtree of node 4

Branch and bound in a nutshell

- Branch and bound creates the enumeration tree
  - one node at a time
  - one branch at a time
- Before branching on a node j, it solves the LP relaxation of the node j's candidate problem
  - **Candidate problem**
    - original problem with variables fixed according to the partial solution $\mathbf{x}^{(j)}$ corresponding to node j
    - finds best completion of partial solution $\mathbf{x}^{(j)}$
- Depending on the solution to the candidate problem, it either
  - terminates node j
  - branches on node j
- We will examine 4 cases

Termination by infeasibility



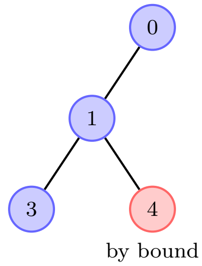
- Node j $\Leftrightarrow$ partial solution $\mathbf{x}^{(j)}$
- Feasible region of candidate problem of node j $\Leftrightarrow$ All possible completions of $\mathbf{x}^{(j)}$ $\Leftrightarrow$ All leaves of node j's subtree
- **Case 1: Termination by infeasibility.** The LP relaxation of the candidate problem of node j is infeasible

$\Rightarrow$ The candidate problem of node j is infeasible

$\Rightarrow$ Any completion of the partial solution $\mathbf{x}^{(j)}$ is infeasible for the original problem!

$\Rightarrow$ Terminate node j (do not branch from node j)

Termination by bound



- Notation:

$$\begin{aligned}\hat{v} &= \text{value of incumbent solution}\\ v^{(j)} &= \text{optimal value of candidate problem for } j\\ \tilde{v}^{(j)} &= \text{ optimal value of LP relaxation}\\ &\qquad\quad \text{of candidate problem for } j\end{aligned}$$

- Recall: candidate problem of j finds best completion of partial solution $\mathbf{x}^{(j)}$

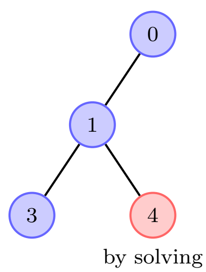
- **Case 2: Termination by bound.** $\tilde{v}^{(j)} \leq \hat{v}$

$\Rightarrow$ $v^{(j)} \leq \tilde{v}^{(j)} \leq \hat{v}$

$\Rightarrow$ No completion of $\mathbf{x}^{(j)}$ is better than the incumbent

$\Rightarrow$ Terminate node j (do not branch from node j)

Termination by solving



- **Case 3: Termination by solving.** $\tilde{v}^{(j)} > \hat{v}$ and the optimal solution $\tilde{\mathbf{x}}^{(j)}$ of the LP relaxation of node j's candidate problem is integer

- $\tilde{\mathbf{x}}^{(j)}$ is integer $\Rightarrow$ $\tilde{\mathbf{x}}^{(j)}$ is optimal for the candidate problem

$\Rightarrow$ $v^{(j)} = \tilde{v}^{(j)} > \hat{v}$

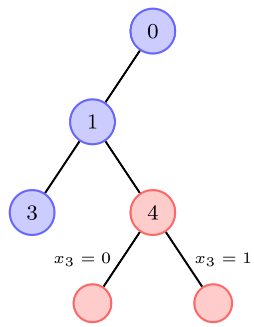
$\Rightarrow$ We have found a feasible solution that is better than the incumbent

$\Rightarrow$ Save solution $\mathbf{x}^{(j)}$ as new incumbent

⇒ No completion of partial solution $\mathbf{x}^{(j)}$ will be better

⇒ Terminate node j (do not branch from node j)

Branching



- **Case 4: Branching.** $\tilde{v}^{(j)} > \hat{v}$ and the optimal solution $\tilde{\mathbf{x}}^{(j)}$ of the LP relaxation of node j's candidate problem is <u>not</u> integer

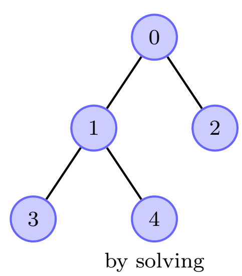
⇒ It is possible that a completion of the partial solution $\mathbf{x}^{(j)}$ may have a better objective value

- Branch at node j: pick some variable that is not fixed in the partial solution $\mathbf{x}^{(j)}$ and create a child node for each possible value

Active nodes

- A node is called **active** if it has been analyzed:
  - it has no children
  - it has not been terminated

- For example:



  The active nodes here are 2 and 3

- Initially, the only active node is the root node 0

- Branch and bound stops when there are no more active nodes

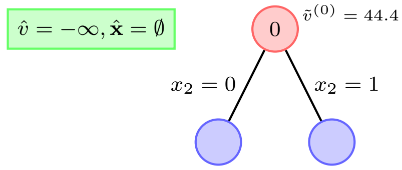
LP-based branch and bound algorithm for 0-1 ILPS

- We have essentially described the whole branch and bound algorithm, piecemeal
- We'll give an abbreviated version of the algorithm
- A = set of active nodes
- $\hat{\mathbf{x}}$ = incumbent solution, $\hat{v}$ = value of incumbent solution
- $\mathrm{LP}^{(t)}$ = LP relaxation of node t's candidate problem
- $\tilde{\mathbf{x}}^{(t)}$ = optimal solution to $\mathrm{LP}^{(t)}$, $\tilde{v}^{(t)}$ = optimal value of $\mathrm{LP}^{(t)}$

0. **Initialize.**
    - $A \leftarrow \{\text{partial solution with no variables fixed}\}$
    - $\hat{\mathbf{x}} \leftarrow \emptyset$, $\hat{v} \leftarrow -\infty$ (or some external heuristic finds an incumbent)
    - Solution counter $t \leftarrow 0$
1. **Select.**
    - If $A = \emptyset$, then $\hat{\mathbf{x}}$ is optimal if it exists, and the problem is infeasible if no incumbent exists
    - Else,
        - remove a node from A
        - label this node t
        - categorize t into one of the four cases

- **Case 1: Termination by infeasibility** $\mathrm{LP}^{(t)}$ is infeasible. Terminate node t.
- **Case 2: Termination by bound** $\tilde{v}^{(t)} \leq \hat{v}$. Terminate node t.
- **Case 3: Termination by solution** $\tilde{v}^{(t)} > \hat{v}$ and $\tilde{\mathbf{x}}^{(t)}$ is integer. Terminate node t, set $\hat{\mathbf{x}} \leftarrow \tilde{\mathbf{x}}^{(t)}$ and $\hat{v} \leftarrow \tilde{v}^{(t)}$
- **Case 4: Branching** $\tilde{v}^{(t)} > \hat{v}$ and $\tilde{\mathbf{x}}^{(t)}$ is not integer. Choose a variable that is not fixed in partial solution $\mathbf{x}^{(t)}$ and branch on all its possible values
- Increment solution counter $t \leftarrow t + 1$, goto Step 1
- Some areas of vagueness:
    - Which active node to choose in Step 1?
        * In principle, can select any active node
        * One potential rule: **depth first search** - select active node with the most components fixed (deepest in tree)
    - Which variable to branch on?
        * In principle, can select any variable not fixed at node's partial solution
        * One potential rule: choose variable whose LP optimal value at that node is fractional and closest to integer

Branch and bound, illustrated

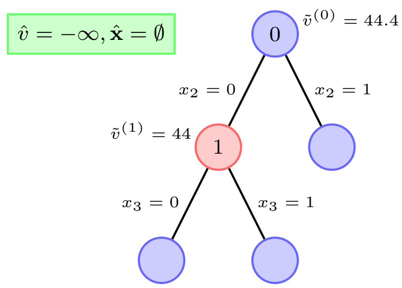


- LP relaxation of candidate problem at root node:

$$\mathrm{LP}^{(0)}: \quad \begin{array}{ll} \max & 16x_1 + 22x_2 + 12x_3 + 8x_4 + 11x_5 + 19x_6 \\ \text{s.t.} & 5x_1 + 7x_2 + 4x_3 + 3x_4 + 4x_5 + 6x_6 \leq 14 \\ & 0 \leq x_i \leq 1 \qquad i = 1, \ldots, 6 \end{array}$$

- Optimal solution: $\tilde{v}^{(0)} = 44.4$, $\tilde{x}^{(0)} = (1, 0.43, 0, 0, 0, 1)$

$\Rightarrow$ Case 4: branch on x_2

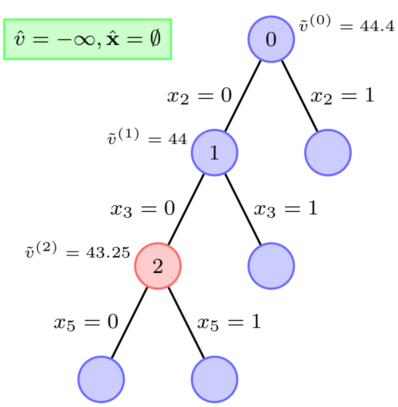


- LP relaxation of candidate problem at root node $\mathrm{LP}^{(1)}$:

$$\begin{array}{ll} \max & 16x_1 + 22x_2 + 12x_3 + 8x_4 + 11x_5 + 19x_6 \\ \text{s.t.} & 5x_1 + 7x_2 + 4x_3 + 3x_4 + 4x_5 + 6x_6 \leq 14 \\ & x_2 = 0 \\ & 0 \leq x_i \leq 1 \qquad i = 1, \ldots, 6 \end{array}$$

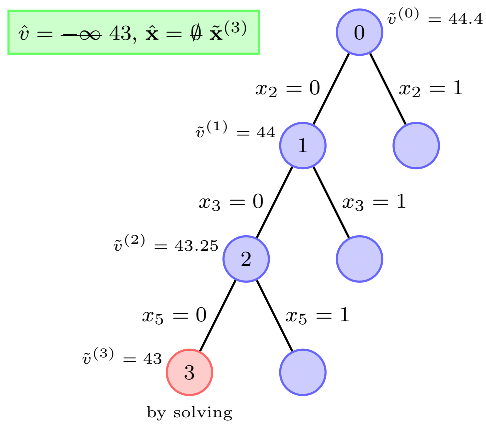
- Optimal solution: $\tilde{v}^{(1)} = 44$, $\tilde{x}^{(1)} = (1, 0, 0.75, 0, 0, 1)$

$\Rightarrow$ Case 4: branch on x_3



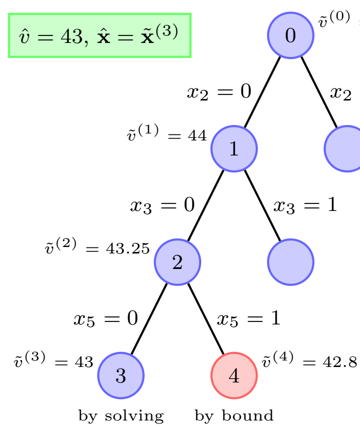
- Solve $\mathrm{LP}^{(2)}$: $\tilde{v}^{(2)} = 43.25$, $\tilde{x}^{(2)} = (1, 0, 0, 0, 0.75, 1)$

$\Rightarrow$ Case 4: branch on x_5



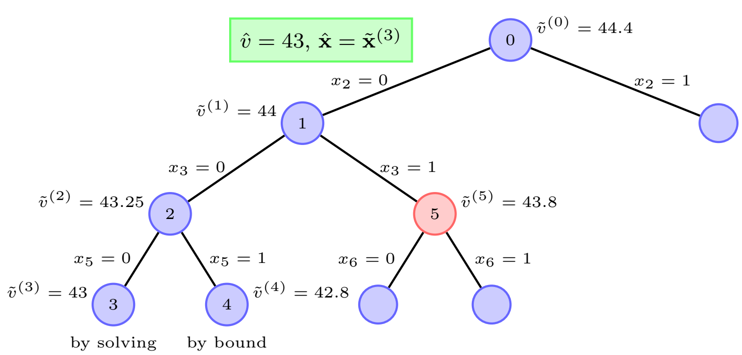
- Solve LP$^{(3)}$: $\tilde{v}^{(3)} = 43$, $\tilde{x}^{(3)} = (1, 0, 0, 1, 0, 1)$
- Solving LP$^{(3)}$ yields integer solution that is better than incumbent

$\Rightarrow$ Case 3: replace incumbent with $\tilde{x}^{(3)}$, terminate node 3



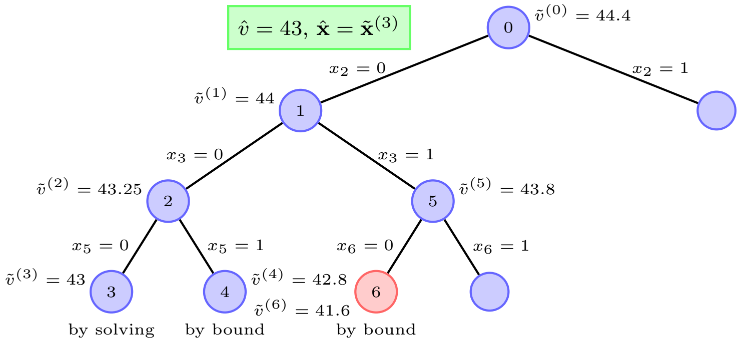
- Solve LP$^{(4)}$: $\tilde{v}^{(4)} = 42.8$, $\tilde{x}^{(4)} = (1, 0, 0, 0, 1, 0.83)$

$\Rightarrow$ Case 2: terminate node 4 by bound



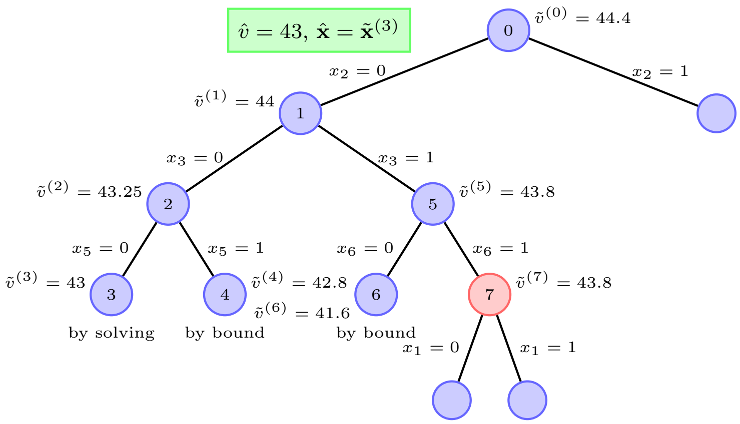
- Solve LP$^{(5)}$: $\tilde{v}^{(5)} = 43.8$, $\tilde{x}^{(5)} = (1, 0, 1, 0, 0, 0.83)$

$\Rightarrow$ Case 4: branch on x_6



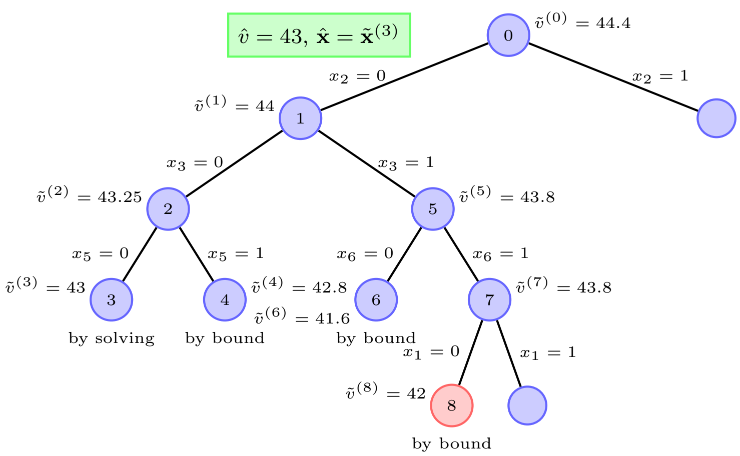
- Solve LP$^{(6)}$: $\tilde{v}^{(6)} = 41.6$, $\tilde{x}^{(6)} = (1, 0, 1, 0.33, 1, 0)$

$\Rightarrow$ Case 2: terminate node 6 by bound



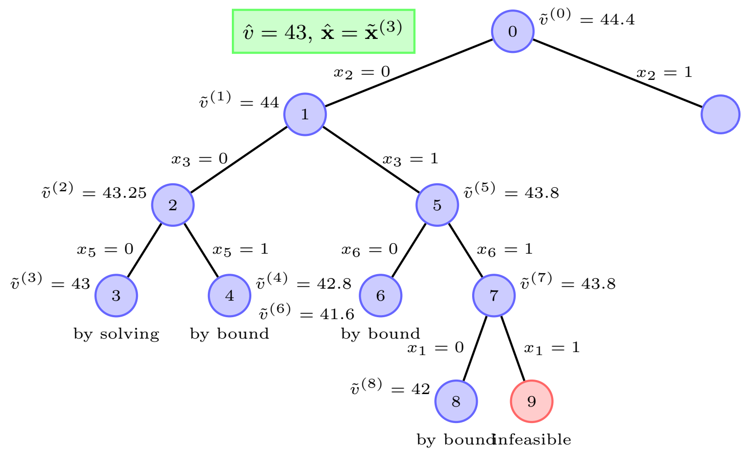
- Solve LP$^{(7)}$: $\tilde{v}^{(7)} = 43.8$, $\tilde{x}^{(7)} = (0.8, 0, 1, 0, 0, 1)$

$\Rightarrow$ Case 4: branch on x_1



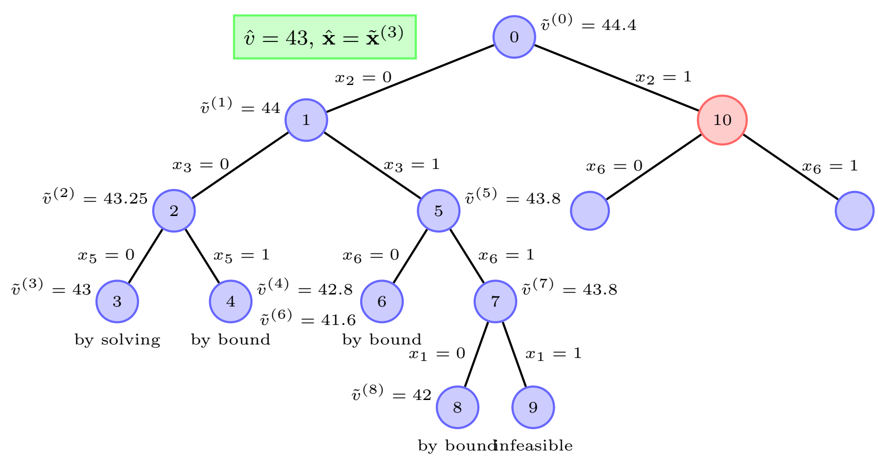
- Solve LP$^{(8)}$: $\tilde{v}^{(8)} = 42$, $\tilde{x}^{(8)} = (0, 0, 1, 0, 1, 1)$

$\Rightarrow$ Case 2: terminate node 8 by bound



- Solve LP$^{(9)}$: infeasible

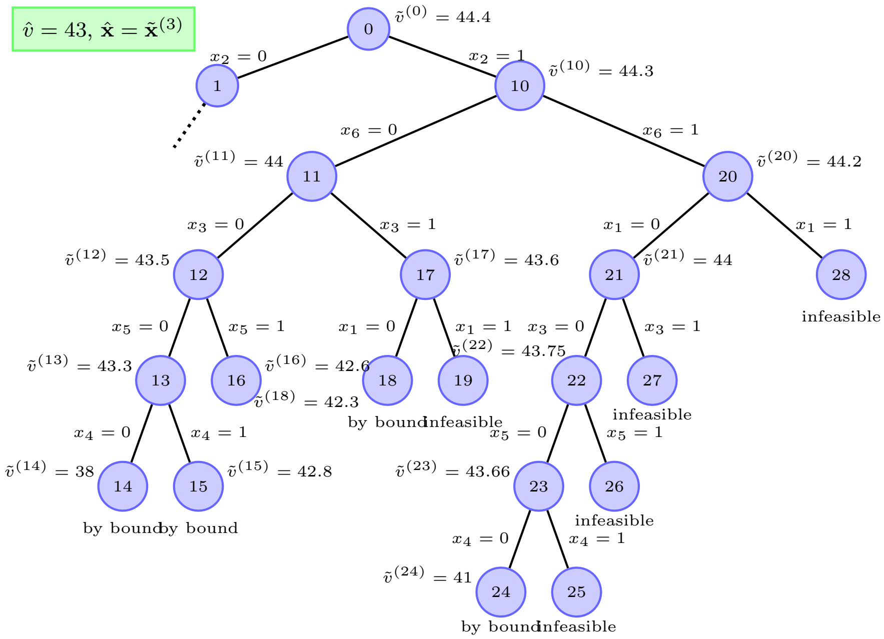
$\Rightarrow$ Case 2: terminate node 9 by infeasibility



- Solve LP$^{(10)}$: $\tilde{v}^{(10)} = 44.3$, $\tilde{x}^{(10)} = (1, 1, 0, 0, 0, 0.33)$

$\Rightarrow$ Case 4: branch on x_6

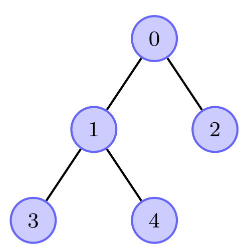
- And we keep on going in a similar manner until there are no active nodes left



- We solved 29 LPs to get an optimal solution to the knapsack problem
- We found the optimal solution at the **third** iteration, but could not conclude that this solution was optimal until the 28th iteration
- What can we say about the quality of the solution we obtained at the third iteration?

Branch and bound family tree terminology

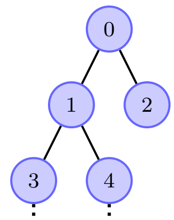
- Easiest to explain by a picture:



- Node 1 is the **parent** of nodes 3 and 4
- Nodes 1 and 2 are the **children** of node 0

Parent bounds

- Suppose we have a maximization integer linear program
- Example:



- $\mathbf{x}^{(j)}$ = partial solution at node j
- $\text{IP}^{(j)}$ = node j's candidate problem
- $\text{LP}^{(j)}$ = LP relaxation of node j's candidate problem
- $v^{(j)}$ = optimal value of $\text{IP}^{(j)}$
- $\tilde{v}^{(j)}$ = optimal value of $\text{LP}^{(j)}$

- $v^{(3)}$ = value of best completion of $\mathbf{x}^{(3)}$
- $\text{LP}^{(3)} = \text{LP}^{(1)}$ + one additional variable fixed $\Rightarrow \tilde{v}^{(3)} \leq \tilde{v}^{(1)}$

$\Rightarrow v^{(3)} \leq \tilde{v}^{(3)} \leq \tilde{v}^{(1)}$

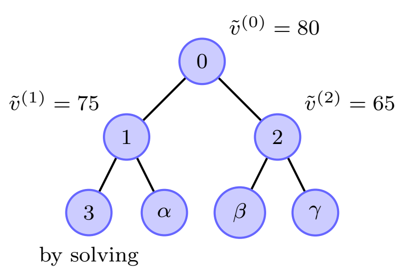
- $\text{LP}^{(1)}$ also provides an upper bound on the value of the best completion of $\mathbf{x}^{(3)}$

Parent bounds

- For maximization ILPs, the optimal value of the LP relaxation of a parent node's candidate problem provides an upper bound on the objective value of any completion of its children
- Similar reasoning for minimization ILPs

Terminating nodes with parent bounds

- Can use parent bounds to terminate some nodes even faster
- Example:



- α, β, and γ are active nodes
- Suppose new incumbent found at node 3 has value $\hat{v} = 70$
- Parent bound: all completions of node 2 have value ≤ 65

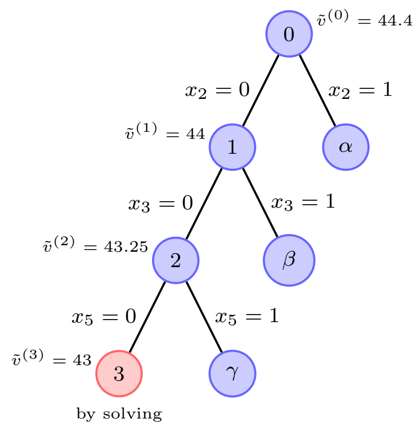
$\Rightarrow$ No point in exploring β, γ, can terminate them immediately

- Whenever branch and bound discovers a new incumbent solution, any active node whose parent bound is no better than the value of the new incumbent solution can be immediately terminated

How good is the current incumbent?

- Sometimes just finding a feasible solution is difficult
- Would be nice to approximate how close a given solution is to optimal
- LP relaxations and parent bounds can help us do this

$$\boxed{\hat{v} = \cancel{-\infty}\ 43,\ \hat{\mathbf{x}} = \cancel{\emptyset}\ \tilde{\mathbf{x}}^{(3)}}$$



- At node 3, we get a new incumbent with value $\hat{v} = 43$
- Any solution that might improve upon the incumbent is a completion of some active partial solution

⇒ Using parent bounds on α, β, γ, we can conclude at this point in branch and bound that the optimal value must be at most

$$\max\{44.4, 44, 43.25\} = 44.4$$

- What if we use the current incumbent as an approximation to the optimal solution?
- The current incumbent is at most

$$\frac{\text{(best possible)} - \text{(best known)}}{\text{best known}} = \frac{44.4 - 43}{43} = 3.25\%$$

below optimal

- For maximization ILPs, we can obtain an upper bound on the optimal value by
  - looking at the parent bound of all active nodes, and
  - taking the highest parent bound
- Can use this to obtain a bound in the error in using the incumbent as an approximation

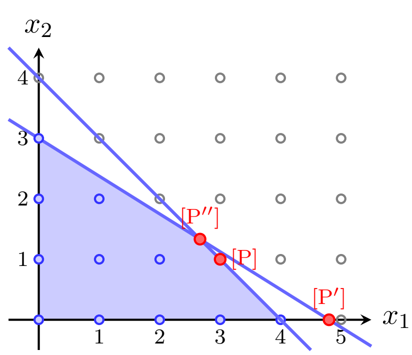
Selecting active nodes

- We used the depth first search rule in our illustration
- Other ideas:
  - **Best first** search selects at each iteration an active node with the best parent bound
  - **Depth forward best back** search selects
    - a deepest active node after a branching
    - an active node with best parent bound after a termination

Branch and cut

$$\begin{array}{rl} \text{[P] max} & 3x_1 + 4x_2 \\ \text{s.t.} & 5x_1 + 8x_2 \leq 24 \\ & x_1, x_2 \geq 0 \\ & x_1, x_2 \text{ integer} \end{array}$$

$$\begin{array}{rl} \text{[P}''\text{] max} & 3x_1 + 4x_2 \\ \text{s.t.} & 5x_1 + 8x_2 \leq 24 \\ & x_1 + x_2 \leq 4 \\ & x_1, x_2 \geq 0 \end{array}$$



- Add constraint $x_1 + x_2 \leq 4$
- Note: this constraint holds for all integer feasible solutions, but cuts off feasible solutions from the LP relaxation [P′]
- The constraint $x_1 + x_2 \leq 4$ is a **valid inequality** for [P]
- **Branch and cut** algorithms modify branch and bound by attempting to strengthen the LP relaxations of the candidate problems by adding valid inequalities
- Important: valid inequalities must hold for all feasible solutions to the full model, not just the candidate problems

- Added valid inequalities should cut off (render infeasible) the optimal solution to the LP relaxations of the candidate problems

- Sophisticated modern ILP codes are typically some variant of branch and cut

Branch and bound

- It is the starting point for all solution techniques for integer programming.
- Lots of research has been carried out over the past 40 years to make it more and more efficient.
- But, it is an art form to make it efficient. (We did get a sense why.)
- Integer programming is intrinsically difficult.
- How to do branching for general integer programs?