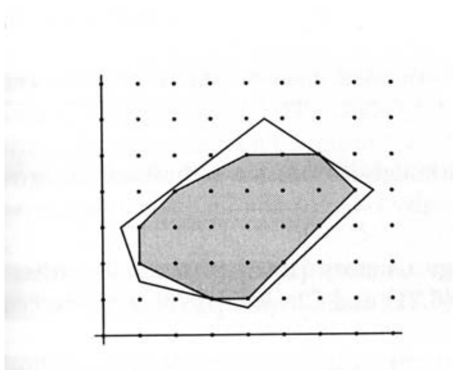
Cutting Plane Methods I

Cutting Planes

- Consider $\max\{wx : Ax \leq b, x \text{ integer}\}$.
- Establishing the optimality of a solution is equivalent to proving $wx \leq t$ is valid for all integral solutions of $Ax \leq b$, where t is the maximum value.
- Without the integrality restriction, we could prove the validity of $wx \leq t$ with the help of LP duality.
- Our goal is to establish a similar method for integral solutions.
- Consider the linear system

$$\begin{aligned} 2x_1 + 3x_2 &\leq 27 \\ 2x_1 - 2x_2 &\leq 7 \\ -6x_1 - 2x_2 &\leq -9 \\ -2x_1 - 6x_2 &\leq -11 \\ -6x_1 + 8x_2 &\leq 21 \end{aligned}$$



- As can easily be seen, every integral solution satisfies $x_2 \leq 5$.
- However, we cannot derive this directly with LP duality because there is a fractional vector, $(9/2, 6)$, with $x_2 = 6$.
- Instead, let us multiply the last inequality by 1/2:

$$-3x_1 + 4x_2 \leq 21/2.$$

- Every integral solution satisfies the stronger inequality

$$-3x_1 + 4x_2 \leq 10,$$

  obtained by rounding 21/2 down to the nearest integer.
- Multiplying this inequality by 2 and the first inequality by 3, and adding the resulting inequalities, gives:

$$17x_2 \leq 101.$$

- Multiplying by 1/17 and rounding down the right-hand side, we can conclude:

$$x_2 \leq 5.$$

- In general, suppose our system consists of

$$a_i x \leq b_i \quad i = 1, \ldots, m.$$

- Let $y_1, \ldots, y_m \geq 0$ and set

$$c = \sum_{i=1}^{m} y_i a_i$$

and

$$d = \sum_{i=1}^{m} y_i b_i.$$

- Trivially, every solution to $Ax \leq b$ satisfies $cx \leq d$.
- If c is integral, all integral solutions to $Ax \leq b$ also satisfy

$$cx \leq \lfloor d \rfloor.$$

- $cx \leq \lfloor d \rfloor$ is called a *Gomory-Chvátal cut* (GC cut).
- "Cut" because the rounding operation cuts off part of the original polyhedron.
- GC cuts can also be defined directly in terms of the polyhedron P defined by $Ax \leq b$: just take a valid inequality $cx \leq d$ for P with c integral and round down to $cx \leq \lfloor d \rfloor$.
- The use of the nonnegative numbers y_i is to provide a derivation of $cx \leq \lfloor d \rfloor$. With the y_i's in hand, we are easily convinced that $cx \leq d$ and $cx \leq \lfloor d \rfloor$ are indeed valid.

Cutting-Plane Proofs

- A *cutting-plane proof* of an inequality $wx \leq t$ from $Ax \leq b$ is a sequence of inequalities

$$a_{m+k} x \leq b_{m+k} \quad k = 1, \ldots, M$$

together with nonnegative numbers

$$y_{ki} \quad k = 1, \ldots, M, i = 1, \ldots, m + k - 1$$

such that for each $k = 1, \ldots, M$, the inequality $a_{m+k} x \leq b_{m+k}$ is derived from

$$a_i x \leq b_i \quad i = 1, \ldots, m + k - 1$$

using the numbers y_{ki}, $i = 1, \ldots, m + k - 1$, and such that the last inequality in the sequence is $wx \leq t$.

Theorem 1 (Chvátal 1973, Gomory 1960)**.** *Let $P = \{x : Ax \leq b\}$ be a rational polytope and let $wx \leq t$ be an inequality, with w integral, satisfied by all integral vectors in P. Then there exists a cutting-plane proof of $wx \leq t'$ from $Ax \leq b$, for some $t' \leq t$.*

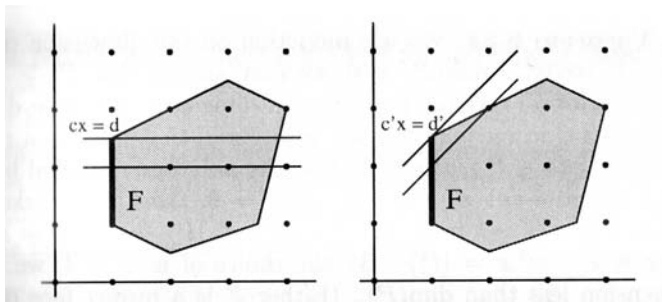
- Proof idea:

- Push $wx \leq l$ into the polytope as far as possible.
- Use induction to show that the face F induced by $wx \leq l$ contains no integral points.
- Push the inequality to $wx \leq l - 1$.
- Continuing this, we eventually reach $wx \leq t$.

- Need technique to translate the cutting-plane proof on F to a proof on the entire polytope:

Lemma 2. *Let F be a face of a rational polytope P. If $cx \leq \lfloor d \rfloor$ is a GC cut for F, then there exists a GC cut $c'x \leq \lfloor d' \rfloor$ for P such that*

$$F \cap \{x : c'x \leq \lfloor d' \rfloor\} = F \cap \{x : cx \leq \lfloor d \rfloor\}.$$



Proof:

- Let $P = \{x : A'x \leq b', A''x \leq b''\}$, where A'' and b'' are integral.
- Let $F = \{x : A'x \leq b', A''x = b''\}$.
- We may assume that $d = \max\{cx : x \in F\}$.
- By LP duality, there exist vectors $y' \geq 0$ and y'' such that

$$\begin{aligned} y'A' + y''A'' &= c \\ y'b' + y''b'' &= d. \end{aligned}$$

- To obtain a GC cut for P we must replace y'' by a vector that is nonnegative.
- To this end, define

$$\begin{aligned} c' &= c - \lfloor y'' \rfloor A'' = y'A' + (y'' - \lfloor y'' \rfloor)A'' \\ d' &= d - \lfloor y'' \rfloor b'' = y'b' + (y'' - \lfloor y'' \rfloor)b'' \end{aligned}$$

- Then c' is integral, and $c'x \leq d'$ is a valid inequality for P.
- Moreover, since $\lfloor d \rfloor = \lfloor d' \rfloor + \lfloor y'' \rfloor b''$,

$$\begin{aligned} F \cap \{x : c'x \leq \lfloor d' \rfloor\} &= \\ F \cap \{x : c'x \leq \lfloor d' \rfloor, \lfloor y'' \rfloor A''x = \lfloor y'' \rfloor b''\} &= \\ F \cap \{x : cx \leq \lfloor d \rfloor\}. \end{aligned}$$

$\square$

Theorem 3. *Let $P = \{x : Ax \leq b\}$ be a rational polytope that contains no integral vectors. Then there exists a cutting-plane proof of $0x \leq -1$ from $Ax \leq b$.*

Proof:

- Induction on the dimension of P.
- Theorem trivial if $\dim(P) = 0$. So assume $\dim(P) \geq 1$.
- Let $wx \leq l$ be an inequality, with w integral, that induces a proper face of P.
- Let $\bar{P} = \{x \in P : wx \leq \lfloor l \rfloor\}$.
- If $\bar{P} = \emptyset$, then we can use Farkas' Lemma to deduce $0x \leq -1$ from $Ax \leq b$, $wx \leq \lfloor l \rfloor$.
- Suppose $\bar{P} \neq \emptyset$, and let $F = \{x \in \bar{P} : wx = \lfloor l \rfloor\}$.
- Note that $\dim(F) < \dim(P)$.
- By the induction hypothesis, there exists a cutting-plane proof of $0x \leq -1$ from $Ax \leq b$, $wx = \lfloor l \rfloor$.
- Using the lemma, we get a cutting-plane proof, from $Ax \leq b$, $wx \leq \lfloor l \rfloor$ of an inequality $cx \leq \lfloor d \rfloor$ such that
$$\bar{P} \cap \{x : cx \leq \lfloor d \rfloor, wx = \lfloor l \rfloor\} = \emptyset.$$
- Thus, after applying this sequence of cuts to $\bar{P}$, we have $wx \leq \lfloor l \rfloor - 1$ as a GC cut.
- As P is bounded, $\min\{wx : x \in P\}$ is finite.
- Continuing in the above manner, letting $\bar{P} = \{x \in P : wx \leq \lfloor l \rfloor - 1\}$, and so on, we eventually obtain a cutting-plane proof of some $wx \leq t$ such that $P \cap \{x : wx \leq t\} = \emptyset$.
- With Farkas' Lemma we then derive $0x \leq -1$ from $Ax \leq b$, $wx \leq t$. □

Theorem 4 (Chvátal 1973, Gomory 1960)**.** *Let $P = \{x : Ax \leq b\}$ be a rational polytope and let $wx \leq t$ be an inequality, with w integral, satisfied by all integral vectors in P. Then there exists a cutting-plane proof of $wx \leq t'$ from $Ax \leq b$, for some $t' \leq t$.*

Proof:

- Let $l = \max\{wx : x \in P\}$, and let $\bar{P} = \{x \in P : wx \leq \lfloor l \rfloor\}$.
- If $\lfloor l \rfloor \leq t$, we are done, so suppose not.
- Consider the face $F = \{x \in \bar{P} : wx = \lfloor l \rfloor\}$.
- Since $t < \lfloor l \rfloor$, F contains no integral points.
- By the previous theorem, there exists a cuting-plane proof of $0x \leq -1$ from $Ax \leq b$, $wx = \lfloor l \rfloor$.

- Using the lemma, we get a cutting-plane proof, from $Ax \leq b$, $wx \leq \lfloor l \rfloor$ of an inequality $cx \leq \lfloor d \rfloor$ such that
$$\bar{P} \cap \{x : cx \leq \lfloor d \rfloor, wx = \lfloor l \rfloor\} = \emptyset.$$
- Thus, after applying this sequence of cuts to $\bar{P}$, we have $wx \leq \lfloor l \rfloor - 1$ as a GC cut.
- Continuing in this fashion, we finally derive an inequality $wx \leq t'$ with $t' \leq t$. □

Chvátal Rank

- GC cuts have an interesting connection with the problem of finding linear descriptions of combinatorial convex hulls.
- In this context, we do not think of cuts coming sequentially, as in cutting-plane proofs, but rather in waves that provide successively tighter approximations to P_I, the convex hull of integral points in P.
- Let P' be the set of all points in P that satisfy every GC cut for P.

Theorem 5 (Schrijver 1980)**.** *If P is a rational polyhedron, then P' is also a rational polyhedron.*

Proof:

- Let $P = \{x : Ax \leq b\}$ with A and b integral.
- Claim: P' is defined by $Ax \leq b$ and all inequalities that can be written as
$$(yA)x \leq \lfloor yb \rfloor$$
for some vector y such that $0 \leq y < 1$ and yA is integral.
- Note that this would give the result.
- So let $wx \leq \lfloor t \rfloor$ be a GC cut, derived from $Ax \leq b$ with the nonnegative vector y.
- Let $y' = y - \lfloor y \rfloor$.
- Then $w' = y'A = w - \lfloor y \rfloor A$ is integral.
- Moreover, $t' = y'b = t - \lfloor y \rfloor b$ differs from t by an integral amount.
- So the cut $w'x \leq \lfloor t' \rfloor$ derived with y', together with the valid inequality $(\lfloor y \rfloor A)x \leq \lfloor y \rfloor b$ sum to $wx \leq t$. □

Letting $P^{(0)} = P$ and $P^{(i)} = (P^{(i-1)})'$, we have
$$P = P^{(0)} \supseteq P^{(1)} \supseteq P^{(2)} \supseteq \cdots \supseteq P_I.$$

Theorem 6. *If P is a rational polyhedron, then $P^{(k)} = P_I$ for some integer k.*

The least k for which $P^{(k)} = P_I$ is called the Chvátal rank of P.

- In general, there is no upper bound on the Chvátal rank in terms of the dimension of the polyhedron.

- For polytopes $P \subseteq [0,1]^n$, the Chvátal rank is $\mathrm{O}(n^2 \log n)$.

- If for a family of polyhedra P the problem $\max\{wx : x \in P_I\}$ is NP-complete, then, assuming NP $\neq$ co-NP, there is no fixed k such that $P^{(k)} = P_I$ for all P.

Gomory's Cutting-Plane Procedure

- Consider $\max\{cx : Ax = b, x \in \mathbb{Z}^n_+\}$.

- Given an (optimal) LP basis B, write the IP as

$$\begin{aligned} \max \quad & c_B B^{-1} b + \sum_{j \in N} \bar{c}_j x_j & \\ \text{s.t.} \quad & x_{B_i} + \sum_{j \in N} \bar{a}_{ij} x_j = \bar{b}_i & i = 1, \ldots, m \\ & x_j \in \mathbb{Z} & j = 1, \ldots, n \end{aligned}$$

- $\bar{c}_j \leq 0$ for all $j \in N$; $\bar{b}_i \geq 0$ for all $i = 1, \ldots, m$.

- If the LP solution is not integral, then there exists row i with $\bar{b}_i \notin \mathbb{Z}$.

- The GC cut for row i is $x_{B_i} + \sum_{j \in N} \lfloor \bar{a}_{ij} \rfloor x_j \leq \lfloor \bar{b}_i \rfloor$.

- Substitute for x_{B_i} to get $\sum_{j \in N} (\bar{a}_{ij} - \lfloor \bar{a}_{ij} \rfloor) x_j \geq \bar{b}_i - \lfloor \bar{b}_i \rfloor$.

- Or if $f_{ij} = \bar{a}_{ij} - \lfloor \bar{a}_{ij} \rfloor$, $f_i = \bar{b}_i - \lfloor \bar{b}_i \rfloor$, then

$$\sum_{j \in N} f_{ij} x_{ij} \geq f_i.$$